ON AUTOMORPHISMS OF LIE ALGEBRAS OF CLASSICAL TYPE. III

BY

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In this paper, the techniques developed in [9; 10] are applied to a study of concrete realizations of the exceptional Lie algebras of classical types F_4 and E_6 (see [8]) as Lie algebras of linear transformations of a split exceptional Jordan algebra $\mathfrak{J}$. Except for fields of characteristic 2 and 3, the interpretations of these Lie algebras given for characteristic zero by Chevalley and Schafer [3] are shown to carry over to arbitrary base fields. When several more low characteristics are exempted from consideration (on the basis of a failure of applicability of certain results on exponentials in [10]), the structure of the automorphism groups of these algebras is here determined in terms of certain linear groups acting in $\mathfrak{J}$. Some of the latter groups have also been studied recently by Jacobson [6; 7], who has proved the simplicity of a group here shown to be isomorphic to the group of invariant automorphisms of the Lie algebra in question (F_4 or E_6). The present paper also demonstrates the simplicity of these groups, by showing that they can be identified with groups whose simplicity has been proved by Chevalley [2].

1. **Derivations of the split exceptional Jordan algebra.** Let $\mathfrak{C}$ be the split Cayley algebra over a field $\mathfrak{F}$ of characteristic $\neq 2$, regarded as all matrices of the form

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix},$$

where $\alpha, \beta \in \mathfrak{F}$ and $a, b \in \mathfrak{V}_3(\mathfrak{F})$, the space of triples of elements of $\mathfrak{F}$, with

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + a\cdot d & \alpha c + \delta a - (b \wedge d) \\ \gamma b + \beta d + (a \wedge c) & \beta\delta + b\cdot c \end{pmatrix},$$

where $x\cdot y$ and $x \wedge y$ are the usual scalar and vector products in $\mathfrak{V}_3(\mathfrak{F})$, and with the involution $x \to \bar{x}$ defined by

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}^- = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix}$$

[10]. Then $\mathfrak{C}$ is an alternative algebra over $\mathfrak{F}$, and the symmetric bilinear form (x, y) defined by $(x, y)1 = (x\bar{y} + y\bar{x})/2 = (\bar{x}y + \bar{y}x)/2$ is nondegenerate on

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$\mathfrak{C}$. If we let $e_1=(1, 0, 0)$, $e_2=(0, 1, 0)$, $e_3=(0, 0, 1)$ be the usual basis for $\mathfrak{B}_3(\mathfrak{F})$ and set

$$u_i = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \qquad u_{4+i} = -2\begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix}, \qquad 1 \leqq i \leqq 3,$$
$$u_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u_8 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

then $u_1, \cdots, u_8$ is a basis for $\mathfrak{C}$, and the matrix for (x, y) relative to this basis is

$$\left(\begin{array}{c|c} 0 & I_4 \\ \hline I_4 & 0 \end{array}\right).$$

Thus the form (x, y) is of maximal Witt index, namely 4. The multiplication table for the basis $u_1, \cdots, u_8$ in $\mathfrak{C}$ is as follows, where first factors stand to the left of rows, second factors at the head of columns:

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	0	$\frac{1}{2}u_7$	$-\frac{1}{2}u_6$	u_1	$-u_8$	0	0	0
u_2	$-\frac{1}{2}u_7$	0	$\frac{1}{2}u_5$	u_2	0	$-u_8$	0	0
u_3	$\frac{1}{2}u_6$	$-\frac{1}{2}u_5$	0	u_3	0	0	$-u_8$	0
u_4	0	0	0	u_4	u_5	u_6	u_7	0
u_5	$-2u_4$	0	0	0	0	$4u_3$	$-4u_2$	$2u_5$
u_6	0	$-2u_4$	0	0	$-4u_3$	0	$4u_1$	$2u_6$
u_7	0	0	$-2u_4$	0	$4u_2$	$-4u_1$	0	$2u_7$
u_8	$2u_1$	$2u_2$	$2u_3$	0	0	0	0	$2u_8$

We shall refer to the *principle of triality*: If T is a linear transformation of $\mathfrak{C}$ which is skew with respect to (x, y), then there are uniquely determined skew transformations T^{ϕ}, T^{ψ} such that for all $x, y \in \mathfrak{C}$,

$$(1) \qquad (xy)T^{\psi} = (xT)y + x(yT^{\phi}).$$

(See, for example, [3; 4; 7], or Chapter IV of [1].)

If E_{ij} are the usual (8 by 8) matrix units relative to the basis $u_1, \cdots, u_8$ of $\mathfrak{C}$, then $H_i = E_{ii} - E_{i+4,i+4}$ is skew, $1 \leqq i \leqq 4$. In the Cartan subalgebra $\mathfrak{H}$ of the Lie algebra $\mathfrak{D}_4$, of all skew transformations of $\mathfrak{C}$, spanned by these H_i, we find that ϕ and ψ map $\mathfrak{H}$ into itself, their matrices relative to the basis $H_1, \cdots, H_4$ of $\mathfrak{H}$ being:

$$\phi: \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix},$$

(2)

$$\psi: \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

That these satisfy the fundamental relation (1) is verified by checking $(u_ju_k)H_i^\psi = (u_jH_i)u_k + u_j(u_kH_i^\phi)$ for all u_j, u_k, H_i.

Now let $\mathfrak{J}$ be the 27-dimensional space over $\mathfrak{F}$ of all 3 by 3 matrices of the form

$$a = \begin{pmatrix} \alpha_{11} & a_{12} & a_{13} \\ \bar{a}_{12} & \alpha_{22} & a_{23} \\ \bar{a}_{13} & \bar{a}_{23} & \alpha_{33} \end{pmatrix},$$

$\alpha_{ii} \in \mathfrak{F}$, $a_{ij} \in \mathfrak{C}$. With the composition $a \cdot b = ab + ba$, $\mathfrak{J}$ is a Jordan algebra, which we shall call *the split exceptional Jordan algebra* [7]. Here if $a = (a_{ij})$ and $b = (b_{ij})$ are matrices with entries in $\mathfrak{C}$, the product $ab = (c_{ij})$ is defined as usual by $c_{ij} = \sum_k a_{ik}b_{kj}$, the multiplication of entries being that in $\mathfrak{C}$.

Let $\mathfrak{L}$ be the Lie algebra of all derivations of $\mathfrak{J}$, i.e., all linear transformations D of $\mathfrak{J}$ over $\mathfrak{F}$ such that $(a \cdot b)D = (aD) \cdot b + a \cdot (bD)$ for all $a, b \in \mathfrak{J}$. Then, as usual, $ID = 0$, where I denotes the identity matrix. Next let $x = \operatorname{diag}\{1, 1, -1\} \in \mathfrak{J}$, and let

$$xD = a = \begin{pmatrix} \alpha_{11} & a_{12} & a_{13} \\ \bar{a}_{12} & \alpha_{22} & a_{23} \\ \bar{a}_{13} & \bar{a}_{23} & \alpha_{33} \end{pmatrix}.$$

For economy of space, write $a = \operatorname{diag}\{\alpha_{11}, \alpha_{22}, \alpha_{33}\} + a_{12}(1, 2) + a_{13}(1, 3) + a_{23}(2, 3)$. From $x \cdot x = I + I$, we have $0 = 2(ID) = 2x \cdot a = 4 \operatorname{diag}\{\alpha_{11}, \alpha_{22}, \alpha_{33}\} + 4a_{12}(1, 2)$, or $\alpha_{11} = \alpha_{22} = \alpha_{33} = 0$, $a_{12} = 0$, and $xD = a_{13}(1, 3) + a_{23}(2, 3)$. Similarly, if $y = \operatorname{diag}\{1, -1, 1\} \in \mathfrak{J}$, $yD = b_{12}(1, 2) + b_{23}(2, 3)$, and $x \cdot y = 2x + 2y - 2I$ yields $(x \cdot y)D = (xD) \cdot y + x \cdot (yD) = 2xD + 2yD$, or $2b_{12}(1, 2) + 2a_{13}(1, 3) = 2b_{12}(1, 2) + 2a_{13}(1, 3) + 2(a_{23} + b_{23})(2, 3)$, so that $b_{23} = -a_{23}$. Writing a_{12} in place of b_{12}, we see from $\operatorname{diag}\{\alpha_{11}, \alpha_{22}, \alpha_{33}\} = (\alpha_{22} + \alpha_{33})I/2 + (\alpha_{11} - \alpha_{22})y/2 + (\alpha_{11} - \alpha_{33})x/2$ that

$$\begin{aligned}&\operatorname{diag}\{\alpha_{11}, \alpha_{22}, \alpha_{33}\}D\\ &= \frac{1}{2}[(\alpha_{11}-\alpha_{22})a_{12}(1,2)+(\alpha_{11}-\alpha_{33})a_{13}(1,3)+(\alpha_{22}-\alpha_{33})a_{23}(2,3)],\end{aligned} \tag{3}$$

where a_{12}, a_{13}, a_{23} are determined as above by xD and yD.

Next let $a\in\mathfrak{C}$, and let $a(1, 2)D=\operatorname{diag}\{\beta_{11}, \beta_{22}, \beta_{33}\}+b_{12}(1, 2)+b_{13}(1, 3)+b_{23}(2, 3)$. Since $a(1, 2)\cdot a(1, 2)=(a, a)(I+x)$, we find by computing $(a(1, 2)\cdot a(1, 2))D$ as $2a(1, 2)\cdot(a(1, 2))D$ and as $(a, a)(I+x)D=(a, a)xD$ by (3), that

$$\beta_{11}+\beta_{22}=0; \qquad (a, a)a_{13}=2ab_{23}; \qquad (a, a)a_{23}=2\bar{a}b_{23}; \qquad (a, b_{12})=0. \tag{4}$$

From $a(1, 2)\cdot y=0$ we obtain $a(1, 2)D\cdot y+a(1, 2)\cdot(yD)=0$, so that one finds

$$\beta_{11}=-(a, a_{12}); \qquad \beta_{33}=0; \qquad b_{13}=\frac{1}{2}aa_{23}. \tag{5}$$

From $a(1, 2)\cdot x=2a(1, 2)$, we find as above that $b_{23}=\bar{a}a_{13}/2$. If we set $2b_{12}=aT$, $T\in\mathfrak{E}(\mathfrak{C})$, the algebra of all linear transformations of $\mathfrak{C}$, we see from the condition $(a, b_{12})=0$ of (4) that $(a, aT)=0$ for all $a\in\mathfrak{C}$, hence that T is skew: $T\in\mathfrak{D}_4$. Thus we have

$$\begin{aligned}&a(1, 2)D\\ &= \frac{1}{2}[(a, a_{12})\operatorname{diag}\{-1, 1, 0\}+(aT)(1, 2)+(aa_{23})(1, 3)+(\bar{a}a_{13})(2, 3)],\end{aligned} \tag{6}$$

where $T\in\mathfrak{D}_4$, and where a_{12}, a_{13}, a_{23} are as in (3). Similarly, we have

$$\begin{aligned}&a(1, 3)D\\ &= \frac{1}{2}[(a, a_{13})\operatorname{diag}\{-1, 0, 1\}-(a\bar{a}_{23})(1, 2)+(aU)(1, 3)+(\bar{a}_{12}a)(2, 3)],\end{aligned} \tag{7}$$

$$\begin{aligned}&a(2, 3)D\\ &= \frac{1}{2}[(a, a_{23})\operatorname{diag}\{0, -1, 1\}-(a_{13}\bar{a})(1, 2)-(a_{12}a)(1, 3)+(aV)(2, 3)]\end{aligned} \tag{8}$$

where U, $V\in\mathfrak{D}_4$. In particular we note that D maps $\mathfrak{J}$ into the subspace $\mathfrak{J}'$ of matrices of trace zero.

Now from $a(1, 2)\cdot b(2, 3)=(ab)(1, 3)$ we obtain $a(1, 2)D\cdot b(2, 3)+a(1, 2)\cdot b(2, 3)D=(ab)(1, 3)D$. Computing and comparing entries in the (1, 3)-positions yields $(ab)U=(aT)b+a(bV)$ for all $a, b\in\mathfrak{C}$. That is, $V=T^{\phi}, U=T^{\psi}$, where ϕ and ψ are the automorphisms of $\mathfrak{D}_4$ given in the principle of triality. Thus D is completely determined by the parameters $a_{12}, a_{13}, a_{23}\in\mathfrak{C}$, and by $T\in\mathfrak{D}_4$. Hence the dimension of $\mathfrak{L}$ does not exceed $3\cdot 8+28=52$.

The mapping $D\rightarrow(a_{12}, a_{13}, a_{23}, T)$ is linear from $\mathfrak{L}$ into $\mathfrak{C}\oplus\mathfrak{C}\oplus\mathfrak{C}\oplus\mathfrak{D}_4$, and is one-one. As in §6 of [10], to show it is onto we need only show that if for $a_{12}, a_{13}, a_{23} \in \mathfrak{C}$, $T \in \mathfrak{D}_4$, we define a linear transformation D of $\mathfrak{J}$ by

$$(\mathrm{diag}\{\beta_{11}, \beta_{22}, \beta_{33}\} + b_{12}(1, 2) + b_{13}(1, 3) + b_{23}(2, 3))D$$

$$\begin{aligned} &= \frac{1}{2}\,[\mathrm{diag}\{-2(b_{12}, a_{12}) - 2(b_{13}, a_{13}), 2(b_{12}, a_{12}) - 2(b_{23}, a_{23}), 2(b_{13}, a_{13}) \\ &\quad + 2(b_{23}, a_{23})\} + ((\beta_{11} - \beta_{22})a_{12} - b_{13}\bar{a}_{23} - a_{13}\bar{b}_{23} + b_{12}T)(1, 2) \\ &\quad + ((\beta_{11} - \beta_{33})a_{13} + b_{12}a_{23} - a_{12}b_{23} + b_{13}T^{\psi})(1, 3) \\ &\quad + ((\beta_{22} - \beta_{33})a_{23} + \bar{b}_{12}a_{13} + \bar{a}_{12}b_{13} + b_{23}T^{\phi})(2, 3)], \end{aligned} \tag{9}$$

then D is a derivation of $\mathfrak{J}$. This follows by a rather lengthy calculation which becomes straightforward when the identities in $\mathfrak{C}$ noted below are used:

$\mathfrak{C}$ is an alternative algebra, i.e., $a(ab) = a^2b$, $(ab)b = ab^2$ for all $a, b \in \mathfrak{C}$. If we define $A(a, b, c) = (ab)c - a(bc)$, it follows by polarization of the above that $A(a, b, c) = -A(b, a, c) = -A(a, c, b)$, so that $A(a, b, c)$ is an alternating function of its arguments. For the scalar product in $\mathfrak{C}$ it follows that $(a, bc) = (a\bar{c}, b) = (\bar{b}a, c)$, and that $2(a, b)c - b(\bar{a}c) - a(\bar{b}c) = (a\bar{b})c - a(\bar{b}c) + (b\bar{a})c - b(\bar{a}c) = A(a, \bar{b}, c) + A(b, \bar{a}, c) = A(a, \bar{b}, c) - A(\bar{a}, b, c) = A(a, (\bar{b}+b), c) - A(a, b, c) - A((\bar{a}+a), b, c) + A(a, b, c) = 0$, since $\bar{b}+b$ and $\bar{a}+a$ are in $\mathfrak{F}\cdot 1$. By a similar argument one obtains the second of the two equations

$$2(a, b)c = a(\bar{b}c) + b(\bar{a}c), \qquad 2(a, b)c = (ca)\bar{b} + (cb)\bar{a}. \tag{10}$$

The two identities that require further discussion are

$$(a\bar{b})T = a(bT^{\phi})^{-} + (aT^{\psi})\bar{b}, \qquad (\bar{b}a)T^{\phi} = (bT)^{-}a + \bar{b}(aT^{\psi}) \tag{11}$$

for all $a, b \in \mathfrak{C}$, and for all $T \in \mathfrak{D}_4$. These are linear in b. First let $b = \lambda 1$, $\lambda \in \mathfrak{F}$; then they become $\lambda aT = \lambda a(1T^{\phi})^{-} + \lambda(aT^{\psi})1$ and $\lambda aT^{\phi} = \lambda(1T)^{-}a + \lambda 1(aT^{\psi})$. From the fact that T and T^{ϕ} are skew, we have $(1, 1T) = 0 = (1, 1T^{\phi})$, or $1(1T)^{-} + (1T)\bar{1} = 0 = 1(1T^{\phi})^{-} + (1T^{\phi})\bar{1}$, i.e., $(1T)^{-} = -1T$, $(1T^{\phi})^{-} = -1T^{\phi}$. Thus the desired identities become $\lambda aT = -\lambda a(1T^{\phi}) + \lambda aT^{\psi}$ and $\lambda aT^{\phi} = -\lambda(1T)a + \lambda aT^{\psi}$. These now follow from $aT^{\psi} = (a1)T^{\psi} = (aT)1 + a(1T^{\phi}) = aT + a(1T^{\phi})$ and $aT^{\psi} = (1a)T^{\psi} = (1T)a + 1(aT^{\phi}) = (1T)a + aT^{\phi}$, respectively.

Next let $b \in \mathfrak{C}'$, the space of elements of trace zero in $\mathfrak{C}$; since the form (x, y) is nondegenerate on $\mathfrak{C}$ and since $\mathfrak{C}'$ is the orthogonal complement of the nonisotropic subspace $\mathfrak{F}1$ with respect to this form, we may assume that b runs through a basis for $\mathfrak{C}'$ consisting of elements b with $(b, b) \neq 0$. It remains only to prove (11) when b is of this form. Then we have $(b, b)1 = b\bar{b} = \bar{b}b = \beta 1 \neq 0$, $\beta \in \mathfrak{F}$, and $0 = \bar{b} + b = (1, b)$, $b^2 = -b\bar{b} = -\beta 1$. Thus we must verify

$$-(ab)T = a(bT^{\phi})^{-} - (aT^{\psi})b, \qquad -(ba)T^{\phi} = (bT)^{-}a - b(aT^{\psi}) \tag{12}$$

for such b. We know that $-\beta aT^{\psi} = (ab^2)T^{\psi} = ((ab)b)T^{\psi} = ((ab)T)b + (ab)(bT^{\phi})$, and that $-\beta aT^{\psi} = (b(ba))T^{\psi} = (bT)(ba) + b((ba)T^{\phi})$. Multiplying the former of these equations on the right by b and the second on the left by b gives

$$(13)\qquad \begin{aligned} -\beta(ab)T + ((ab)(bT^{\phi}))b &= -\beta(aT^{\psi})b, \\ b((bT)(ba)) - \beta(ba)T^{\phi} &= -\beta b(aT^{\psi}). \end{aligned}$$

Now $(ab)(bT^{\phi}) = A(a, b, bT^{\phi}) + a(b(bT^{\phi})) = -A(a, bT^{\phi}, b) - a(\bar{b}(bT^{\phi})) = A(a, (bT^{\phi})^{-}, b) - a(2(b, bT^{\phi}) - (bT^{\phi})^{-}b)$, and $(bT)(ba) = -A(bT, b, a) + ((bT)b)a = -A(b, (bT)^{-}, a) - (2(bT, b) - b(bT)^{-})a$. Since T and T^{ϕ} are skew, $(b, bT) = 0 = (b, bT^{\phi})$, and we have $(ab)(bT^{\phi}) = (a(bT^{\phi})^{-})b - a((bT^{\phi})^{-}b) + a((bT^{\phi})^{-}b) = (a(bT^{\phi})^{-})b$, $(bT)(ba) = -(b(bT)^{-})a + b((bT)^{-}a) + (b(bT)^{-})a = b((bT)^{-}a)$; substituting in (13) gives $-\beta(ab)T + ((a(bT^{\phi})^{-})b)b = -\beta(aT^{\psi})b$ and $b(b((bT)^{-}a)) - \beta(ba)T^{\phi} = -\beta b(aT^{\psi})$. From the alternative identities, $-\beta(ab)T - \beta a(bT^{\phi})^{-} = -\beta(aT^{\psi})b$ and $-\beta(bT)^{-}a - \beta(ba)T^{\phi} = -\beta b(aT^{\psi})$. The identities (12) are now immediate.

Thus we can identify $\mathfrak{L}$ with $\mathfrak{C} \oplus \mathfrak{C} \oplus \mathfrak{C} \oplus \mathfrak{D}_4$ as above. Let $D = (a_{12}, a_{13}, a_{23}, T)$, $E = (b_{12}, b_{13}, b_{23}, U)$ be derivations of $\mathfrak{J}$, in this identification. Computing the effect of $[DE]$ on x y, and $a(1, 2)$ as above yields for $[DE] = (c_{12}, c_{13}, c_{23}, V)$,

$$(14)\qquad \begin{aligned} c_{12} &= \frac{1}{2}(a_{12}U - b_{12}T - a_{13}\bar{b}_{23} + b_{13}\bar{a}_{23}), \\ c_{13} &= \frac{1}{2}(a_{13}U^{\psi} - b_{13}T^{\psi} - b_{12}a_{23} + a_{12}b_{23}), \\ c_{23} &= \frac{1}{2}(a_{23}U^{\phi} - b_{23}T^{\phi} + \bar{b}_{12}a_{13} - \bar{a}_{12}b_{13}), \\ aV &= \frac{1}{2}(4(a, b_{12})a_{12} - 4(a, a_{12})b_{12} - (aa_{23})\bar{b}_{23} + (ab_{23})\bar{a}_{23} \\ &\quad - b_{13}(\bar{a}_{13}a) + a_{13}(\bar{b}_{13}a) + a[TU]), \qquad a \in \mathfrak{C}. \end{aligned}$$

The subspace $\mathfrak{H}$ of $\mathfrak{L}$ spanned by the derivations $2(0, 0, 0, H_i) = h_i$, $1 \leqq i \leqq 4$, is thus a commutative subalgebra of $\mathfrak{L}$. From our computation of H_i^{ϕ}, H_i^{ψ}, it follows that $\mathfrak{H}$ is also spanned by the elements $2(0, 0, 0, H_i^{\phi})$ and is spanned by the $2(0, 0, 0, H_i^{\psi})$. Moreover we have

$$[(u_j, 0, 0, 0), h_i] = (u_jH_i, 0, 0, 0) = \begin{cases} \delta_{ij}(u_j, 0, 0, 0), & 1 \leqq j \leqq 4, \\ -\delta_{i,j-4}(u_j, 0, 0, 0), & 5 \leqq j \leqq 8. \end{cases}$$

If we define $\beta_j \in \mathfrak{H}^*$, $1 \leqq j \leqq 4$, by $\beta_j(h_i) = \delta_{ij}$, $1 \leqq i \leqq 4$, then for each $h \in \mathfrak{H}$ we have

$$(15)\qquad \begin{aligned} [(u_j, 0, 0, 0), h] &= \beta_j(h)(u_j, 0, 0, 0), & 1 \leqq j \leqq 4, \\ [(u_{j+4}, 0, 0, 0), h] &= -\beta_j(h)(u_{j+4}, 0, 0, 0), & 1 \leqq j \leqq 4. \end{aligned}$$

We also have $[(0, u_j, 0, 0), h_i] = (0, u_j H_i^{\psi}, 0, 0)$, $[(0, 0, u_j, 0), h_i] = (0, 0, u_j H_i^{\phi}, 0)$, $1 \leqq j \leqq 8$, so that if we define $\beta_5, \cdots, \beta_{12} \in \mathfrak{H}^*$ by $\beta_{j+4}(h_i) = (-1/2) + \delta_{ij}$, $1 \leqq j \leqq 4$, $1 \leqq i \leqq 4$; $\beta_{j+8}(h_i) = (-1/2) + \delta_{ij}$, $1 \leqq i, j \leqq 3$, $\beta_{j+8}(h_4) = 1/2$, $1 \leqq j \leqq 3$, $\beta_{12}(h_i) = -1/2$, $1 \leqq i \leqq 4$, then it follows by (2) that for all $h \in \mathfrak{H}$,

$$\begin{aligned} [(0, u_j, 0, 0), h] &= \beta_{j+4}(h)(0, u_j, 0, 0), \\ [(0, u_{j+4}, 0, 0), h] &= -\beta_{j+4}(h)(0, u_{j+4}, 0, 0), \qquad 1 \leqq j \leqq 4; \end{aligned} \tag{16}$$

$$\begin{aligned} [(0, 0, u_j, 0), h] &= \beta_{j+8}(h)(0, 0, u_j, 0), \\ [(0, 0, u_{j+4}, 0), h] &= -\beta_{j+8}(h)(0, 0, u_{j+4}, 0), \qquad 1 \leqq j \leqq 4. \end{aligned} \tag{17}$$

From observation of the formula for V in (14), we find $\beta_{13}, \cdots, \beta_{24}$ in $\mathfrak{H}^*$ and 24 nonzero elements $T \in \mathfrak{D}_4$ such that for all $h \in \mathfrak{H}$,

$$[(0, 0, 0, T), h] = \beta(h)(0, 0, 0, T), \tag{18}$$

where, for the various values of T, β runs through $\pm\beta_{13}, \cdots, \pm\beta_{24}$. If $13 \leqq j \leqq 18$, we have $\beta_j(h_i) = \delta_{mi} - \delta_{ni}$, $1 \leqq m < n \leqq 4$; for $19 \leqq j \leqq 24$, $\beta_j(h_i) = -\delta_{mi} - \delta_{ni}$, $1 \leqq m < n \leqq 4$, $1 \leqq i \leqq 4$. Here T runs through the transformations $E_{ij} - E_{j+4,i+4}$, $1 \leqq i, j \leqq 4$; $E_{i,j+4} - E_{j,i+4}$; $E_{i+4,j} - E_{j+4,i}$, $1 \leqq i < j \leqq 4$, where the E_{mn} are 8 by 8 matrix units relative to the basis $u_1, \cdots, u_8$ of $\mathfrak{C}$. For example, with $T = E_{16} - E_{25}$, $(0, 0, 0, T)$ belongs to the function β with $\beta(h_i) = -\delta_{1i} - \delta_{2i}$, and $(0, 0, 0, E_{52} - E_{61})$ to the function $-\beta$.

We assume now that the characteristic of $\mathfrak{F}$ is neither 2 nor 3. The $\pm\beta_i$, $1 \leqq i \leqq 24$, are 48 distinct nonzero linear functions on $\mathfrak{H}$ such that for each β, there is a nonzero $e_\beta \in \mathfrak{L}$ such that $[e_\beta h] = \beta(h) e_\beta$ for all $h \in \mathfrak{H}$. If we set $\mathfrak{L}_\beta = \mathfrak{F} e_\beta$ for each such β, then the $\mathfrak{L}_\beta$ are independent, and the $\mathfrak{L}_\beta$, together with $\mathfrak{H}$, are also independent. Thus $\mathfrak{L} = \mathfrak{H} + \sum_\beta \mathfrak{L}_\beta$, a vector space direct sum. It follows as in §6 of [10] that $\mathfrak{H}$ is an abelian Cartan subalgebra of $\mathfrak{L}$, with respect to which the $\pm\beta_i$ are roots. The roots β span $\mathfrak{H}^*$, since $\beta_1, \cdots, \beta_4$ already do so. Now it follows as in §6 of [10] that the center of $\mathfrak{L}$ is zero. Evidently $\mathfrak{L}_\beta \subseteq [\mathfrak{L}\mathfrak{L}]$ for each root β; also $\mathfrak{H} \subseteq [\mathfrak{L}\mathfrak{L}]$, since $\mathfrak{H}$ is spanned by the $[e_{-\beta} e_\beta]$ for β among $\beta_{13}, \cdots, \beta_{24}$. Thus $\mathfrak{L}$ is seen to satisfy the postulates of [8], except possibly for (iv) and (v). For β among $\pm\beta_{13}, \cdots, \pm\beta_{18}$, the fact that $[\mathfrak{L}_{-\beta}\mathfrak{L}_\beta] \neq 0$ follows from (14), (18), and $[E_{ij} - E_{j+4,i+4}, E_{ji} - E_{i+4,j+4}] = E_{ii} - E_{jj} + E_{j+4,j+4} - E_{i+4,i+4}$, $1 \leqq i, j \leqq 4$; for β among $\pm\beta_{19}, \cdots, \pm\beta_{24}$, it follows from (14), (18), and $[E_{i,j+4} - E_{j,i+4}, E_{j+4,i} - E_{i+4,j}] = E_{ii} + E_{jj} - E_{i+4,i+4} - E_{j+4,j+4}$, $1 \leqq i, j \leqq 4$, $i \neq j$.

For β among $\pm\beta_1, \cdots, \pm\beta_4$, it follows from (14) that

$$[(u_i, 0, 0, 0), (u_{i+4}, 0, 0, 0)] = (0, 0, 0, V), \qquad 1 \leqq i \leqq 4,$$

where $aV = 2(a, u_{i+4})u_i - 2(a, u_i)u_{i+4}$; thus $u_i V = 2u_i \neq 0$, and $[\mathfrak{L}_{-\beta}\mathfrak{L}_\beta] \neq 0$. For β among $\pm\beta_5, \cdots, \pm\beta_8$, we have for $1 \leqq i \leqq 4$, $[(0, u_i, 0, 0), (0, u_{i+4}, 0, 0)] = (0, 0, 0, V)$, where $aV = (u_i(\bar{u}_{i+4}a)/2 - u_{i+4}(\bar{u}_i a))$. If $1 \leqq i \leqq 3$, it follows from the table on page 287 and the fact that $\bar{u}_i = -u_i$, $\bar{u}_{i+4} = -u_{i+4}$, that $u_8 V = -u_i u_{i+4} = u_8$, so $V \neq 0$. If $i=4$, then $\bar{u}_4 = u_8/2$, $\bar{u}_8 = 2u_4$, and $u_1 V = -u_8 u_1/2 = -u_1$, so $V \neq 0$. Thus $[\mathfrak{L}_{-\beta}\mathfrak{L}_\beta] \neq 0$ for these values of β. A similar argument applies for β among $\pm\beta_9, \cdots, \pm\beta_{12}$. Hence all $[\mathfrak{L}_{-\beta}\mathfrak{L}_\beta]$ are one-dimensional, and (iv) of [8] is satisfied.

Now the roots α_1, defined as the root to which $(0, 0, 0, E_{32}-E_{67})$ belongs; α_2, the root to which $(0, 0, 0, E_{16}-E_{25})$ belongs; $\alpha_3 = \beta_1$; $\alpha_4 = -\beta_5$ form a basis for $\mathfrak{H}^*$, and the totality of roots are: $\pm\alpha_i$, $1 \leqq i \leqq 4$; $\pm(\alpha_i+\alpha_{i+1})$, $1 \leqq i \leqq 3$; $\pm(\alpha_i+\alpha_{i+1}+\alpha_{i+2})$, $i=1, 2$; $\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$; $\pm(\alpha_2+2\alpha_3)$; $\pm(\alpha_1+\alpha_2+2\alpha_3)$; $\pm(\alpha_2+2\alpha_3+\alpha_4)$; $\pm(\alpha_2+2\alpha_3+2\alpha_4)$; $\pm(\alpha_1+2\alpha_2+2\alpha_3)$; $\pm(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$; $\pm(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)$; $\pm(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)$; $\pm(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)$; $\pm(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)$; $\pm(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)$; $\pm(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)$; $\pm(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)$; $\pm(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)$. Since the characteristic of $\mathfrak{F}$ is not 2 or 3, it follows by inspection of this list that not all $\alpha+k\beta$ can be roots (or zero) for any nonzero root β, and also that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ form a fundamental system of roots relative to $\mathfrak{H}$ in the sense of [8]. This system is of type F_4, as is seen from the above. Thus all axioms (i)–(v) of [8] are satisfied, and by Theorem 8 of [8], $\mathfrak{L}$ is (normal) simple and is a representative of the isomorphism class F_4 of [8]. In combination with Theorem 9 of [8], this reasoning gives the following theorem:

THEOREM 1. *The Lie algebra $\mathfrak{L}$ of derivations of the split exceptional Jordan algebra $\mathfrak{J}$ is a simple Lie algebra of classical type F_4 over the field $\mathfrak{F}$ of characteristic $\neq 2$, 3. Every Lie algebra of classical type over $\mathfrak{F}$ with a fundamental system of roots of type F_4 is isomorphic to $\mathfrak{L}$.*

2. **The enveloping algebra of $\mathfrak{L}$ in $\mathfrak{J}'$.** If we let $u_1, \cdots, u_8$ be our basis for $\mathfrak{C}$, then every element of $\mathfrak{D}_4$, acting in $\mathfrak{C}$, has as matrix relative to this basis a matrix of the form

$$X = \left(\begin{array}{c|c} (\xi) & (\eta) \\ \hline (\zeta) & -(\xi)' \end{array}\right),$$

where (ξ), (η), (ζ) are 4 by 4 matrices with $(\eta)' = -(\eta)$, $(\zeta)' = -(\zeta)$. One verifies directly that the images of this element under the triality automorphisms ϕ, ψ are

$$X^\phi = \left(\begin{array}{c|c} (\alpha) & (\beta) \\ \hline (\gamma) & -(\alpha)' \end{array}\right),$$

$$X^\psi = \left(\begin{array}{c|c} (\mu) & (\nu) \\ \hline (\rho) & -(\mu)' \end{array}\right),$$

where

$$(\alpha)=\begin{pmatrix} \frac{1}{2}(\xi_{11}-\xi_{22}-\xi_{33}+\xi_{44}) & \xi_{12} & \xi_{13} & -2\eta_{14} \\ \xi_{21} & \frac{1}{2}(-\xi_{11}+\xi_{22}-\xi_{33}+\xi_{44}) & \xi_{23} & -2\eta_{24} \\ \xi_{31} & \xi_{32} & \frac{1}{2}(-\xi_{11}-\xi_{22}+\xi_{33}+\xi_{44}) & -2\eta_{34} \\ \frac{1}{2}\zeta_{14} & \frac{1}{2}\zeta_{24} & \frac{1}{2}\zeta_{34} & -\frac{1}{2}(\xi_{11}+\xi_{22}+\xi_{33}+\xi_{44}) \end{pmatrix},$$

$$(\beta)=\begin{pmatrix} 0 & -\frac{1}{2}\xi_{43} & \frac{1}{2}\xi_{42} & -\frac{1}{4}\zeta_{23} \\ \frac{1}{2}\xi_{43} & 0 & -\frac{1}{2}\xi_{41} & \frac{1}{4}\zeta_{13} \\ -\frac{1}{2}\xi_{42} & \frac{1}{2}\xi_{41} & 0 & -\frac{1}{4}\zeta_{12} \\ \frac{1}{4}\zeta_{23} & -\frac{1}{4}\zeta_{13} & \frac{1}{4}\zeta_{12} & 0 \end{pmatrix}, \quad (\gamma)=\begin{pmatrix} 0 & 2\xi_{34} & -2\xi_{24} & -4\eta_{23} \\ -2\xi_{34} & 0 & 2\xi_{14} & 4\eta_{13} \\ 2\xi_{24} & -2\xi_{14} & 0 & -4\eta_{12} \\ 4\eta_{23} & -4\eta_{13} & 4\eta_{12} & 0 \end{pmatrix},$$

(19)

$$(\mu)=\begin{pmatrix} \frac{1}{2}(\xi_{11}-\xi_{22}-\xi_{33}-\xi_{44}) & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \frac{1}{2}(-\xi_{11}+\xi_{22}-\xi_{33}-\xi_{44}) & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \frac{1}{2}(-\xi_{11}-\xi_{22}+\xi_{33}-\xi_{44}) & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \frac{1}{2}(-\xi_{11}-\xi_{22}-\xi_{33}+\xi_{44}) \end{pmatrix},$$

$$(\nu)=\begin{pmatrix} 0 & -\frac{1}{4}\zeta_{34} & \frac{1}{4}\zeta_{24} & -\frac{1}{4}\zeta_{23} \\ \frac{1}{4}\zeta_{34} & 0 & -\frac{1}{4}\zeta_{14} & \frac{1}{4}\zeta_{13} \\ -\frac{1}{4}\zeta_{24} & \frac{1}{4}\zeta_{14} & 0 & -\frac{1}{4}\zeta_{12} \\ \frac{1}{4}\zeta_{23} & -\frac{1}{4}\zeta_{13} & \frac{1}{4}\zeta_{12} & 0 \end{pmatrix}, \quad (\rho)=\begin{pmatrix} 0 & -4\eta_{34} & 4\eta_{24} & -4\eta_{23} \\ 4\eta_{34} & 0 & -4\eta_{14} & 4\eta_{13} \\ -4\eta_{24} & 4\eta_{14} & 0 & -4\eta_{12} \\ 4\eta_{23} & -4\eta_{13} & 4\eta_{12} & 0 \end{pmatrix}.$$

Using the uniqueness of X^ϕ and X^ψ, these assertions are readily checked by seeing that for each pair of basis elements u_i, u_j, $(u_iu_j)X^\psi=(u_iX)u_j+u_i(u_jX^\phi)$.

Let $w_1=\operatorname{diag}\{1,-1,0\}$, $w_2=\operatorname{diag}\{0,1,-1\}$. Then w_1, w_2 and the $a(j,k)$, $1\leqq j<k\leqq 3$, $a\in\mathfrak{C}$, span the space $\mathfrak{J}'$. Since $\mathfrak{L}$ maps $\mathfrak{J}'$ into $\mathfrak{J}'$ and I into 0, we may identify $\mathfrak{L}$ with its restriction to $\mathfrak{J}'$. In this sense, we have

LEMMA 1. $\mathfrak{L}^4=\mathfrak{E}(\mathfrak{J}')$, *the full algebra of linear transformations of* $\mathfrak{J}'$.

To prove the lemma, we first observe the effect on a general element of $\mathfrak{J}'$ of our basis for $\mathfrak{L}$. Thus if $a=\alpha_1w_1+\alpha_2w_2+a_{12}(1,2)+a_{13}(1,3)+a_{23}(2,3)$, we have

$$
(20)\qquad
\begin{aligned}
a(u_i, 0, 0, 0) &= -(a_{12}, u_i)w_1 + \frac{1}{2}(2\alpha_1 - \alpha_2)u_i(1, 2) - \frac{1}{2}(u_i a_{23})(1, 3) \\
&\quad + \frac{1}{2}(\bar{u}_i a_{13})(2, 3); \\
a(0, u_i, 0, 0) &= -(a_{13}, u_i)(w_1 + w_2) - \frac{1}{2}(u_i \bar{a}_{23})(1, 2) + \frac{1}{2}(\alpha_1 + \alpha_2)u_i(1, 3) \\
&\quad + \frac{1}{2}(\bar{a}_{12} u_i)(2, 3); \\
a(0, 0, u_i, 0) &= -(a_{23}, u_i)w_2 - \frac{1}{2}(a_{13}\bar{u}_i)(1, 2) + \frac{1}{2}(a_{12}u_i)(1, 3) \\
&\quad + \frac{1}{2}(2\alpha_2 - \alpha_1)u_i(2, 3); \qquad 1 \leqq i \leqq 8,
\end{aligned}
$$

and

$$
a(0, 0, 0, T) = \frac{1}{2}(a_{12}T)(1, 2) + \frac{1}{2}(a_{13}T^{\psi})(1, 3) + \frac{1}{2}(a_{23}T^{\psi})(2, 3),\ T \in \mathfrak{D}_4.
$$

From the first three of (20), we see that $a(u_i, 0, 0, 0)^2 = -(a_{12}, u_i)u_i(1, 2)$, $a(0, u_i, 0, 0)^2 = -(a_{13}, u_i)u_i(1, 3)$, $a(0, 0, u_i, 0)^2 = -(a_{23}, u_i)u_i(2, 3)$, $1 \leqq i \leqq 8$. Thus if we denote by $E(v, w)$ the matrix unit sending the basis vector v into the vector w of the same fixed basis, and all other basis vectors into 0, it follows that $\mathfrak{L}^2$ contains

$$
(21)\qquad E(u_i(j, k), u_{i+4}(j, k)); \qquad E(u_{i+4}(j, k), u_i(j, k)), \qquad 1 \leqq j < k \leqq 3,\ 1 \leqq i \leqq 4.
$$

Now $a(u_i, 0, 0, 0)^2(u_j, 0, 0, 0) = (a_{12}, u_i)(u_i, u_j)w_1$, which yields for $j=4$ or $j=i-4$ the result that $\mathfrak{L}^3$ contains each $E(u_i(1, 2), w_1)$, $1 \leqq i \leqq 8$. This and the analogue for $(0, u_i, 0, 0)^2(0, u_j, 0, 0)$ and for $(0, 0, u_i, 0)^2(0, 0, u_j, 0)$ show that $\mathfrak{L}^3$ contains all

$$
(22)\qquad E(u_i(1, 2), w_1),\quad E(u_i(1, 3), w_1) + E(u_i(1, 3), w_2),\quad E(u_i(2, 3), w_2), \qquad 1 \leqq i \leqq 8.
$$

For $1 \leqq i \leqq 4$, we denote by $C(i)$ the set of all $j \neq i$, $1 \leqq j \leqq 4$, and the integer $i+4$, and by $D(i)$ the complement of $C(i)$ in $1, 2, \cdots, 8$. For $5 \leqq i \leqq 8$, we set $C(i) = D(i-4)$, $D(i) = C(i-4)$. Let $C'(i) = \{C(i), 8\} - \{4\}$, $1 \leqq i \leqq 3$, $C'(4) = \{1, 2, 3, 4\}$, and let $D'(i)$ be the complement of $C'(i)$, $1 \leqq i \leqq 4$; for $5 \leqq i \leqq 8$, set $C'(i) = D'(i-4)$, $D'(i) = C'(i-4)$. Then left multiplication by u_i, $i \neq 4, 8$, maps the subspace of $\mathfrak{C}$ spanned by the u_j, $j \in C(i)$, onto the subspace spanned by the u_j, $j \in D(i)$, and the latter subspace onto 0. Let multiplication by

u_i, $i=4$, 8, map the subspace spanned by the u_j, $j\in C(i)$, onto 0, and the subspace spanned by the u_j, $j\in D(i)$, onto itself; moreover vectors of this basis are sent onto scalar multiples of vectors of this basis for every i, $1\leqq i\leqq 8$. Similarly, right multiplication by u_i, $i\neq 4$, 8, sends the span of the u_j, $j\in C'(i)$, onto the span of the u_j, $j\in D'(i)$, and the latter onto zero; for $i=4$, 8, the span of the u_j, $j\in C'(i)$, is mapped onto itself and the remaining u_j go into zero.

Now computing $(u_i, 0, 0, 0)^2(0, u_j, 0, 0)$, we see that a goes into $-(a_{12}, u_i)(\bar{u}_i u_j)(2, 3)/2$. Since the first factor $(u_i, 0, 0, 0)^2$ is essentially a matrix unit $E(u_{i\pm 4}(1, 2), u_i(1, 2))$, we see that for $i\neq 4$, 8, where $\bar{u}_i=-u_i$, our transformation is 0 if $j\in D(i)$, and for $j\in C(i)$ yields in $\mathfrak{L}^3$ the matrix units $E(u_{i\pm 4}(1, 2), u_k(2, 3))$, $k\in D(i)$, i.e., the $E(u_i(1, 2), u_k(2, 3))$, $k\in C(i)$. For $i=4$, 8, $\bar{u}_i=\lambda u_{i\pm 4}$, $0\neq\lambda\in\mathfrak{F}$, yielding in $\mathfrak{L}^3$ all $E(u_{i\pm 4}(1, 2), u_k(2, 3))$, $k\in D(i\pm 4)$; thus $\mathfrak{L}^3$ contains all

$$E(u_i(1, 2), u_j(2, 3)),\quad j\in C(i),\qquad 1\leqq i\leqq 8,\ i\neq 4, 8;$$
$$E(u_i(1, 2), u_j(2, 3)),\quad j\in D(i),\qquad i=4, 8. \tag{23}$$

Similarly from $(u_i, 0, 0, 0)^2(0, 0, u_j, 0)\in\mathfrak{L}^3$, we have in $\mathfrak{L}^3$

$$E(u_i(1, 2), u_j(1, 3)),\quad j\in C(i),\qquad 1\leqq i\leqq 8. \tag{24}$$

From $(0, u_i, 0, 0)^2(u_j, 0, 0, 0)$, $(0, u_i, 0, 0)^2(0, 0, u_j, 0)$, $(0, 0, u_i, 0)^2(u_j, 0, 0, 0)$, $(0, 0, u_i, 0)^2(0, u_j, 0, 0)$, we have in $\mathfrak{L}^3$:

$$\begin{aligned}
&E(u_i(1, 3), u_j(2, 3)), && j\in C'(i), && i\neq 4, 8;\\
&E(u_i(1, 3), u_j(2, 3)), && j\in D'(i), && i=4, 8;\\
&E(u_i(1, 3), u_j(1, 2)), && j\in C(i), && 1\leqq i\leqq 8;\\
&E(u_i(2, 3), u_j(1, 3)), && j\in C'(i), && i\neq 4, 8;\\
&E(u_i(2, 3), u_j(1, 3)), && j\in D'(i), && i=4, 8;\\
&E(u_i(2, 3), u_j(1, 2)), && j\in C'(i), && 1\leqq i\leqq 8.
\end{aligned} \tag{25}$$

From following $(u_i, 0, 0, 0)^2$, $(0, u_i, 0, 0)^2$, $(0, 0, u_i, 0)^2$ by suitable $(0, 0, 0, T)$, where T is selected from the basis for $\mathfrak{D}_4$ used in §1, we find that $\mathfrak{L}^3$ also contains all

$$E(u_i(j, k), u_m(j, k)),\qquad m\neq i,\ 1\leqq i, m\leqq 8,\ 1\leqq j<k\leqq 3. \tag{26}$$

Furthermore, from $(u_{i\pm 4}, 0, 0, 0)(u_i, 0, 0, 0)^2$, $(0, u_{i\pm 4}, 0, 0)(0, u_i, 0, 0)^2$, $(0, 0, u_{i\pm 4}, 0)(0, 0, u_i, 0)^2$, we find in $\mathfrak{L}^3$ the following:

$$\begin{gathered}
2E(w_1, u_i(1, 2)) - E(w_2, u_i(1, 2)),\quad E(w_1, u_i(1, 3)) + E(w_2, u_i(1, 3)),\\
E(w_1, u_i(2, 3)) - 2E(w_2, u_i)(2, 3)),\qquad 1\leqq i\leqq 8.
\end{gathered} \tag{27}$$

Now we show that $E(v, w)\in\mathfrak{L}^4$ for all basis vectors v, w for $\mathfrak{J}'$. Since $E(u_i(j, k), u_{i\pm 4}(j, k))\in\mathfrak{L}^2$ by (21), it follows that $E(u_i(j, k), u_i(j, k))\in\mathfrak{L}^4$;

combined with (26) and the fact that $\mathfrak{L} = [\mathfrak{L}\mathfrak{L}] \subseteq \mathfrak{L}^2$, this yields $E(u_i(j, k), u_m(j, k)) \in \mathfrak{L}^4$ for all i, m, $1 \leqq i, m \leqq 8$, and for $1 \leqq j < k \leqq 3$. Following the elements (23), (24), (25) by suitable $(0, 0, 0, T)$, $T \in \mathfrak{D}_4$, yields in $\mathfrak{L}^4$ all $E(u_i(j, k), u_m(j', k'))$, $1 \leqq i, m \leqq 8$, $1 \leqq j < k \leqq 3$, $1 \leqq j' < k' \leqq 3$. Next, following an $E(u_i(1, 2), u_j(1, 3)) \in \mathfrak{L}^3$ by $(0, u_{j\pm 4}, 0, 0)$ gives $E(u_i(1, 2), w_1) + E(u_i(1, 2), w_2) \in \mathfrak{L}^4$. Since $E(u_i(1, 2), w_1) \in \mathfrak{L}^3 \subseteq \mathfrak{L}^4$ by (22), we have $E(u_i(1, 2), w_2) \in \mathfrak{L}^4$. Similarly, all $E(u_i(1, 3), w_1)$ and all $E(u_i(1, 3), w_2)$ are in $\mathfrak{L}^4$, as well as all $E(u_i(2, 3), w_1)$; all $E(u_i(2, 3), w_2)$ are in $\mathfrak{L}^3 \subseteq \mathfrak{L}^4$.

It remains to show that $E(w_m, v) \in \mathfrak{L}^4$, $m = 1, 2$, for all basis vectors v. by using the fact that for each k, there are an i and a j such that $u_k(2, 3) = \lambda u_i(1, 2)(0, u_j, 0, 0)$, $\lambda \in \mathfrak{F}$, we can follow $2E(w_1, u_i(1, 2)) - E(w_2, u_i(1, 2)) \in \mathfrak{L}^3$ by $\lambda^{-1}(0, u_j, 0, 0)$ to obtain $2E(w_1, u_k(2, 3)) - E(w_2, u_k(2, 3)) \in \mathfrak{L}^4$ for all k. Combined with (27), this yields $E(w_n, u_i(2, 3)) \in \mathfrak{L}^4$, $n = 1, 2$, $1 \leqq i \leqq 8$. Similarly we see that all $E(w_n, u_i(j, k)) \in \mathfrak{L}^4$. Following the first element in (27) by $(u_{i\pm 4}, 0, 0, 0)$, the second by $(0, u_{i\pm 4}, 0, 0)$, the third by $(0, 0, u_{i\pm 4}, 0)$ yields, respectively, $2E(w_1, w_1) - E(w_2, w_1)$, $E(w_1, w_2) + E(w_2, w_2) + E(w_1, w_1) + E(w_2, w_1)$, $E(w_1, w_2) - 2E(w_2, w_2) \in \mathfrak{L}^4$. Finally, computing $(u_1, 0, 0, 0) \cdot (0, u_2, 0, 0)(0, 0, u_3, 0) \in \mathfrak{L}^3 \subseteq \mathfrak{L}^4$ yields in $\mathfrak{L}^4$ a linear combination of matrix units other than $E(w_i, w_j)$ together with a term in $2E(w_1, w_2) - E(w_2, w_2)$; together with the above, this shows all $E(w_i, w_j) \in \mathfrak{L}^4$, $i, j = 1, 2$, and completes the proof of the lemma.

Now we know that if $D \in \mathfrak{L}_\alpha$ for some $\alpha \neq 0$, then $(\text{ad } D)^3 = 0$; combining this fact with the result $\mathfrak{L}^4 = \mathfrak{E}(\mathfrak{J}')$ of Lemma 1 and with Lemma 1 of [10], we know that since D is nilpotent (see [10]), $D^9 = D^{4(3-1)+1} = 0$ on $\mathfrak{J}'$. Since $ID = 0$ we have $D^9 = 0$ for all elements D of all root-spaces $\mathfrak{L}_\alpha$, $\alpha \neq 0$, relative to standard Cartan subalgebras of $\mathfrak{L}$. We shall need the results of §1 of [10] which are valid when $D^{2^{-1}(p+1)} = 0$, where p is the characteristic of $\mathfrak{F}$ (if $p \neq 0$). Thus we assume in the sequel that the characteristic of $\mathfrak{F}$ is either zero or a prime $p > 13$.

3. **Automorphisms of the derivation algebra.** Now the mappings $\exp(D)$ of $\mathfrak{J}$, for D as in the last paragraph, are automorphisms of $\mathfrak{J}$, and for $X \in \mathfrak{L}$, $X \exp(\text{ad}(D)) = (\exp D)^{-1} X (\exp D) = A^{-1} X A$, A an automorphism of $\mathfrak{J}$ [10]. The group $\mathscr{I}(\mathfrak{L})$ of invariant automorphisms [9; 10] is generated by such mappings $X \to A^{-1} X A$, where A is an automorphism of $\mathfrak{J}$. Conversely, if A is an automorphism of $\mathfrak{J}$, the mapping $X \to A^{-1} X A$ is an automorphism of $\mathfrak{L}$. If A is an automorphism of $\mathfrak{J}$ then $IA = I$, and $\mathfrak{J}\mathfrak{L}A = \mathfrak{J}'A$, since $\mathfrak{J}\mathfrak{L} \subseteq \mathfrak{J}'$ and $\mathfrak{J}' = \mathfrak{J}'\mathfrak{L}^4$; but $\mathfrak{J}\mathfrak{L}A = \mathfrak{J}(A\mathfrak{L}A^{-1})A = \mathfrak{J}A\mathfrak{L}$; since $\mathfrak{J}A\mathfrak{L} \subseteq \mathfrak{J}'$, we have $\mathfrak{J}'A \subseteq \mathfrak{J}'$, so that $\mathfrak{J}'A = \mathfrak{J}'$. Thus if $X = A^{-1} X A$ for $A \in \mathfrak{A}(\mathfrak{J})$, the group of automorphisms of $\mathfrak{J}$, and for all $X \in \mathfrak{L}$, we see from Lemma 1 that $A \mid \mathfrak{J}'$ is scalar, so that $IA = I$, $xA = \lambda x$ for all $x \in \mathfrak{J}'$. Thus $(w_1 \cdot w_2)A = w_1 A \cdot w_2 A = \lambda^2 w_1 \cdot w_2$. But $w_1 \cdot w_2 = 2(w_1 - w_2 - I)/3$, so $(w_1 \cdot w_2)A = 2(\lambda w_1 - \lambda w_2 - I)/3 = 2\lambda^2(w_1 - w_2 - I)/3$; hence $\lambda^2 = \lambda$ and $\lambda^2 = 1$; thus $\lambda = 1$, and $A = I$.

Hence the mapping sending $A \in \mathfrak{A}(\mathfrak{J})$ onto the automorphism $I_A: X$

$\rightarrow A^{-1}XA$ of $\mathfrak{L}$ is one-one and an isomorphism of $\mathfrak{A}(\mathfrak{J})$ into $\mathfrak{A}(\mathfrak{L})$. In the algebraically closed case, every automorphism of $\mathfrak{L}$ is an invariant automorphism [9], hence of the form $X \rightarrow A^{-1}XA$, $A \in \mathfrak{A}(\mathfrak{J})$, and the mapping $A \rightarrow I_A$ is an isomorphism of $\mathfrak{A}(\mathfrak{J})$ onto $\mathfrak{A}(\mathfrak{L})$. As in the case of G_2 in [10] we can show that this result remains valid even if $\mathfrak{F}$ is not algebraically closed. Namely:

If $\mathfrak{K}$ is the algebraic closure of $\mathfrak{F}$, then $\mathfrak{J}_{\mathfrak{K}}$ is the split exceptional Jordan algebra over $\mathfrak{K}$, and $\mathfrak{L}_{\mathfrak{K}}$ is the derivation algebra of $\mathfrak{J}_{\mathfrak{K}}$. If σ is an automorphism of $\mathfrak{L}$, σ has a unique extension to an automorphism of $\mathfrak{L}_{\mathfrak{K}}$; hence we have, for this extension σ, $X^\sigma = A^{-1}XA$ for some $A \in \mathfrak{A}(\mathfrak{J}_{\mathfrak{K}})$, all $X \in \mathfrak{L}_{\mathfrak{K}}$. In particular, this relation holds for all $X \in \mathfrak{L}$. Thus there is a nonsingular linear transformation B of $\mathfrak{J}$ such that $X^\sigma = B^{-1}XB$ for all $X \in \mathfrak{L}$ [11]. Now B maps $\mathfrak{J}'$ into $\mathfrak{J}'$, since $\mathfrak{J}'\mathfrak{L} = \mathfrak{J}' = \mathfrak{J}\mathfrak{L}$ implies $\mathfrak{J}'B = \mathfrak{J}'\mathfrak{L}B = \mathfrak{J}'B(B^{-1}\mathfrak{L}B) = \mathfrak{J}'B\mathfrak{L} \subseteq \mathfrak{J}\mathfrak{L} = \mathfrak{J}'$. Since $\mathfrak{L}^4 = \mathfrak{E}(\mathfrak{J}')$, $(\mathfrak{L}_{\mathfrak{K}})^4 = \mathfrak{E}(\mathfrak{J}'_{\mathfrak{K}})$; now $A^{-1}B_{\mathfrak{K}}$ maps $\mathfrak{J}'_{\mathfrak{K}}$ into $\mathfrak{J}'_{\mathfrak{K}}$ and commutes elementwise with $\mathfrak{L}_{\mathfrak{K}}$, so that $A^{-1}B_{\mathfrak{K}} = \lambda I$ on $\mathfrak{J}'_{\mathfrak{K}}$, $\lambda \in \mathfrak{K}$, or $B_{\mathfrak{K}} = \lambda A$, $0 \neq \lambda \in \mathfrak{K}$. But $u_7(1, 3)B/2 = (u_1(1, 2) \cdot u_2(2, 3))B = \lambda(u_1(1, 2) \cdot u_2(2, 3))A$ $= \lambda(u_1(1, 2)A) \cdot (u_2(2, 3)A) = \lambda^{-1}(u_1(1, 2)B) \cdot (u_2(2, 3)B)$, so that $\lambda \in \mathfrak{F}$, $A | \mathfrak{J}' \in \mathfrak{E}(\mathfrak{J}')$, and $A = C_{\mathfrak{K}}$, $C \in \mathfrak{A}(\mathfrak{J})$. Thus we have $X^\sigma = C^{-1}XC$, $C \in \mathfrak{A}(\mathfrak{J})$. As above $C^{-1}XC = X$ for all $X \in \mathfrak{L}$ only if $C = I$. Thus the mapping $A \rightarrow I_A$ is again an isomorphism of $\mathfrak{A}(\mathfrak{J})$ onto $\mathfrak{A}(\mathfrak{L})$.

THEOREM 2. *Except when $\mathfrak{F}$ is of prime characteristic between* 2 *and* 13, *the mapping $A \rightarrow I_A$ which assigns to the automorphism A of the split exceptional Jordan algebra $\mathfrak{J}$ over $\mathfrak{F}$ the automorphism $X \rightarrow A^{-1}XA$ of the Lie algebra $\mathfrak{L}$ of derivations of $\mathfrak{J}$ is an isomorphism of the full automorphism group $\mathfrak{A}(\mathfrak{J})$ of $\mathfrak{J}$ onto the full automorphism group of $\mathfrak{L}$.*

4. **Invariant automorphisms and the group of Chevalley.** The group G' of Chevalley formed over $\mathfrak{F}$ from a complex Lie algebra of type F_4 may be regarded as acting in $\mathfrak{L}$ via the mappings $\exp(\lambda\, \mathrm{ad}(X))$, where X runs through a complete set of root-vectors relative to a fixed standard Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$, the fundamental system of roots relative to $\mathfrak{H}$ being of type F_4 (see [2; 10]). By Theorem 9 of [8], there is an automorphism σ of $\mathfrak{L}$ mapping the Cartan subalgebra $\mathfrak{H}_1$ of $\mathfrak{L}$ spanned by the h_i, $1 \leqq i \leqq 4$, onto $\mathfrak{H}$, and the root-vectors relative to $\mathfrak{H}_1$, as listed above, onto those relative to $\mathfrak{H}$. We have $X^\sigma = A^{-1}XA$, $A \in \mathfrak{A}(\mathfrak{J})$. Now $(\mathrm{ad}(X^\sigma))^3 = 0$ and $(X^\sigma)^9 = 0$ by §2, where X runs through the root-vectors relative to $\mathfrak{H}_1$, so that $Y \exp(\lambda\, \mathrm{ad}(X^\sigma))$ $= (\exp(\lambda X^\sigma))^{-1} Y(\exp(\lambda X^\sigma))$ for all $Y \in \mathfrak{L}$, and G' is the isomorphic image under $B \rightarrow I_B$ of the subgroup $\mathfrak{G}$ of $\mathfrak{A}(\mathfrak{J})$ generated by all $\exp(\lambda X^\sigma)$. The group $A\mathfrak{G}A^{-1}$ is the subgroup of $\mathfrak{A}(\mathfrak{J})$ generated by all $\exp(\lambda X)$, where X runs through the root-vectors relative to $\mathfrak{H}_1$. We show $A\mathfrak{G}A^{-1} = \mathfrak{A}(\mathfrak{J})$. Then it follows that $\mathfrak{G} = \mathfrak{A}(\mathfrak{J})$, and that $G' = \mathfrak{A}(\mathfrak{L}) \cong \mathfrak{A}(\mathfrak{J})$. Since the group $\mathscr{I}(\mathfrak{L})$ of invariant automorphisms of $\mathfrak{L}$ contains G', it follows that $G' = \mathscr{I}(\mathfrak{L}) = \mathfrak{A}(\mathfrak{L})$ $\cong \mathfrak{A}(\mathfrak{J})$. (The fact that $\mathfrak{A}(\mathfrak{J})$ is simple, proved by Jacobson [7], coupled with the fact that $\mathscr{I}(\mathfrak{L})$ is a normal subgroup of $\mathfrak{A}(\mathfrak{L})$, shows that $\mathscr{I}(\mathfrak{L}) = \mathfrak{A}(\mathfrak{L})$ $\simeq \mathfrak{A}(\mathfrak{J})$, but our result is somewhat sharper and, combined with Chevalley's

proof of the simplicity of G', gives another proof of the simplicity of $\mathfrak{A}(\mathfrak{J})$.)

We first investigate the effects of all $\exp(\lambda e_\beta)$, where e_β runs through $2(u_i, 0, 0, 0)$, $2(0, u_i, 0, 0)$, $2(0, 0, u_i, 0)$, and $2(0, 0, 0, T)$, belonging to the roots $\pm\beta_1, \cdots, \pm\beta_{24}$ relative to $\mathfrak{H}_1$: With $b = \operatorname{diag}\{\beta_1, \beta_2, \beta_3\} + b_{12}(1, 2) + b_{13}(1, 3) + b_{23}(2, 3)$, we have

$$
(28)\quad
\begin{aligned}
b\exp(2\lambda(u_i, 0, 0, 0)) &= b + \lambda[\operatorname{diag}\{-2(b_{12}, u_i), 2(b_{12}, u_i), 0\} \\
&\quad + (\beta_1 - \beta_2)u_i(1, 2) - (u_i b_{23})(1, 3) + (\bar{u}_i b_{13})(2, 3)] \\
&\quad - 2\lambda^2(b_{12}, u_i)u_i(1, 2); \\
b\exp(2\lambda(0, u_i, 0, 0)) &= b + \lambda[\operatorname{diag}\{-2(b_{13}, u_i), 0, 2(b_{13}, u_i)\} \\
&\quad - (u_i\bar{b}_{23})(1, 2) + (\beta_1 - \beta_3)u_i(1, 3) + (\bar{b}_{12}u_i)(2, 3)] \\
&\quad - 2\lambda^2(b_{13}, u_i)u_i(1, 3); \\
b\exp(2\lambda(0, 0, u_i, 0)) &= b + \lambda[\operatorname{diag}\{0, -2(b_{23}, u_i), 2(b_{23}, u_i)\} \\
&\quad - (b_{13}\bar{u}_i)(1, 2) + (b_{12}u_i)(1, 3) + (\beta_2 - \beta_3)u_i(2, 3)] \\
&\quad - 2\lambda^2(b_{23}, u_i)u_i(2, 3); \\
b\exp(2\lambda(0, 0, 0, T)) &= b + \lambda[(b_{12}T)(1, 2) + (b_{13}T^\psi)(1, 3) + (b_{23}T^\phi)(2, 3)].
\end{aligned}
$$

Now let $B \in \mathfrak{A}(\mathfrak{J})$. We show that by a succession of right multiplications of B by transformations (28), B can be converted into the identity. Since the transformations (28) generate $A\mathfrak{G}A^{-1}$, it will follow that $B \in A\mathfrak{G}A^{-1}$, so that $A\mathfrak{G}A^{-1} = \mathfrak{A}(\mathfrak{J})$, as asserted.

Now B sends $u_1(1, 2)$ into $u_1(1, 2)B = c = \operatorname{diag}\{\gamma_{11}, \gamma_{22}, \gamma_{33}\} + c_{12}(1, 2) + c_{13}(1, 3) + c_{23}(2, 3)$. We first show that when c_{12} is written in terms of the basis $\{u_i\}$ of $\mathfrak{C}$, we may assume that the coefficient α_1 of u_1 is 1. Suppose first that $\alpha_1 \neq 0$. Then choose λ so that, if α_2 is the coefficient of u_2 in c_{12}, $\lambda\alpha_1 + \alpha_2 \neq 0$, and follow B by $\exp(2\lambda(0, 0, 0, E_{12} - E_{65}))$; the new automorphism sends $u_1(1, 2)$ into an element of $\mathfrak{J}$ having in the $(1, 2)$-position $\alpha_1 u_1 + (\alpha_2 + \lambda\alpha_1)u_2 + \cdots$, $\alpha_2 + \lambda\alpha_1 \neq 0$, and is in $A\mathfrak{G}A^{-1}$ if and only if B is; thus we may assume $\alpha_2 \neq 0$ in B. But then if we follow B by $\exp(2(1-\alpha_1)\alpha_2^{-1}(0, 0, 0, E_{21} - E_{56}))$ we obtain in the $(1, 2)$-position $(\alpha_1 + (1-\alpha_1)\alpha_2^{-1}\alpha_2)u_1 + \cdots = u_1 + \cdots$, i.e., $\alpha_1 = 1$ as desired. Thus we may assume c_{12} has the desired form unless $\alpha_1 = \alpha_2 = 0$. If α_i, the coefficient of u_i in c_{12}, is nonzero for i among 3, 4, 6, 7, 8, a similar procedure can be used to make $\alpha_1 = 1$. If all these are zero, and if $\alpha_5 \neq 0$, a combination of two of the types $\exp(2\lambda(0, 0, 0, T))$ can be used to make $\alpha_1 = 1$. Thus we need only treat the case where $c_{12} = 0$. If $c_{13} \neq 0$ then for some i, $c_{13}\bar{u}_i \neq 0$, so that following B by $\exp(2(0, 0, u_i, 0))$ makes $c_{12} \neq 0$, and we can proceed as before to make $\alpha_1 = 1$. Similarly, for $c_{23} \neq 0$ we use a suitable $\exp(2(0, u_i, 0, 0))$. Thus we may assume $c_{12} = c_{13} = c_{23} = 0$. Now, not both of $\gamma_{11} - \gamma_{22}$, $\gamma_{11} - \gamma_{33}$ are zero, since this implies $\gamma_{11} = \gamma_{22} = \gamma_{33}$, and $u_1(1, 2)B$ has trace $3\gamma_{11} \neq 0$ unless $u_1(1, 2)B = 0$. But $u_1(1, 2)B \in \mathfrak{J}'$, so that $u_1(1, 2)B = 0$, a contradiction to the assumption that B was an automorphism. Thus if

$\gamma_{11}-\gamma_{22}\neq 0$, we follow B by $\exp(2(u_i, 0, 0, 0))$ for any i to obtain $c_{12}\neq 0$; if $\gamma_{11}-\gamma_{22}=0$ then $\gamma_{11}-\gamma_{33}\neq 0$, and following B by $\exp(2(0, u_i, 0, 0))$ makes $c_{13}\neq 0$. In either case the reduction to $\alpha_1=1$ can now be made as before. Thus we may assume $\alpha_1=1$ in B.

Next we show that we may assume $u_1(1, 2)B=u_1(1, 2)$. First of all, follow B by $\exp(2\lambda_i(0, u_i, 0, 0))$, $i\in C(1)$, adding to c_{12} the element $-\lambda_i u_i \bar{c}_{23}$, in which the coefficient of u_1 is zero. Thus $\alpha_1=1$ stays fixed, and suitable choice of the λ_i makes the coefficient of u_j in c_{23} into 0 for all $j\in D(1)$: $c_{23}\rightarrow c_{23}+\lambda_i\bar{c}_{12}u_i$. Similarly, follow the new B by $\exp(2\lambda_i(0, 0, u_i, 0))$, $i\in C(1)$; then the form of c_{23} obtained above is unchanged, as is the property $\alpha_1=1$, and with suitable choice of the λ_i, the coefficient of u_j in c_{13} is made into zero for each $j\in D(1)$. Thus we may assume c_{13} and c_{23} have these forms, and that $\alpha_1=1$. Now operate with $\exp(2\lambda(0, 0, 0, T))$, $T=E_{12}-E_{65}$, $E_{13}-E_{75}$, $E_{14}-E_{85}$; the first two of these T's are fixed under ϕ and ψ, and $(E_{14}-E_{85})^{\psi}=E_{14}-E_{85}$, $(E_{14}-E_{85})^{\phi}=2(E_{63}-E_{72})$. The properties above are unchanged, and the coefficients of u_2, u_3, u_4 in c_{12} are replaced by 0.

Now apply $\exp(2\lambda(0, 0, 0, T))$, $T=E_{16}-E_{25}$, $E_{17}-E_{35}$, $E_{18}-E_{45}$; by checking T^{ϕ} and T^{ψ} from (19), we see that all our earlier properties of B are unchanged, and one obtains $c_{12}=u_1+\alpha_5u_5$, $c_{13}=\sum_{i\in C(1)}\beta_iu_i$, $c_{23}=\sum_{i\in C(1)}\gamma_iu_i$. We may thus assume that B has this form. Now apply $\exp(2\lambda(u_5, 0, 0, 0))$, leaving the above properties fixed, and altering the diagonal by the subtraction of a nonzero multiple of w_1, if $\lambda\neq 0$. Since $u_1(1, 2)\in\mathfrak{J}'$, we have $\operatorname{diag}\{\gamma_{11}, \gamma_{22}, \gamma_{33}\}=\delta_1w_1+\delta_2w_2$, and suitable choice of λ makes $\delta_1=0$. Thus we may assume $c=u_1(1, 2)B=\delta_2w_2+(u_1+\alpha_5u_5)(1, 2)+\sum_{i\in C(1)}\beta_iu_i(1, 3)+\sum_{i\in C(1)}\gamma_iu_i(2, 3)$. From $u_1(1, 2)\cdot u_1(1, 2)=0$ we have $c\cdot c=0$; now $c\cdot c$ has in its (1, 2)-position $2\delta_2(u_1+\alpha_5u_5)+2c_{13}\bar{c}_{23}$, and the coefficient of u_1 in $c_{13}\bar{c}_{23}$ is 0; thus $\delta_2=0$. Then the entry in the (1, 3)-position in $c\cdot c$ is $0=2c_{12}c_{23}$, or $0=u_1c_{23}+\alpha_5u_5c_{23}=\gamma_2u_7/2+\gamma_4u_1-\gamma_3u_6/2-\gamma_5u_8+\sum_{i\in C(1)}\kappa_iu_i$, from which $\gamma_2=\gamma_3=\gamma_4=\gamma_5=0$ and $c_{23}=0$. Likewise, from the fact that the element in the (2, 3)-position in $c\cdot c$ is zero it follows that $c_{13}=0$, and $c=(u_1+\alpha_5u_5)(1, 2)$, $0=c\cdot c=4\operatorname{diag}\{\alpha_5, \alpha_5, 0\}$, and $\alpha_5=0$, $u_1(1, 2)B=c=u_1(1, 2)$. We assume this in the sequel.

Next let $u_1(1, 3)B=c$ as above; from $u_1(1, 2)\cdot u_1(1, 3)=0$, we have $0=(u_1(1, 2)B)\cdot c=u_1(1, 2)\cdot c$. It follows that $\alpha_5=0=\delta_2$, $c_{13}=\sum_{i\in D(1)}\beta_iu_i$, $c_{23}=\sum_{i\in D(1)}\gamma_iu_i$, where the notation is as above. Now we show that we may assume $\beta_1=1$. If β_6, β_7, or β_8 is nonzero, follow B by $\exp(2\lambda(0, 0, 0, T))$, for $T=E_{34}-E_{87}$, $E_{24}-E_{86}$, $E_{27}-E_{36}$, respectively, with suitable λ, to make $\beta_1=1$, leaving $u_1(1, 2)$ fixed. Thus our result is achieved unless $\beta_6=\beta_7=\beta_8=0$. If this is the case, and if $\beta_1\neq 0$, follow B by $\exp(2\lambda(0, 0, 0, E_{74}-E_{83}))$ to make $\beta_6\neq 0$, still leaving $u_1(1, 2)$ fixed; then proceed as above. Thus we may assume either $\beta_1=1$ or $c_{13}=0$. In the latter case, if $\gamma_i\neq 0$, $i\neq 8$, follow B by an $\exp(2(u_j, 0, 0, 0))$, $j\in D(1)$, $j\neq 1$. Then $u_1(1, 2)$ is fixed, but the new c_{13} is $-\sum_{i\in D(1)}\gamma_iu_ju_i\neq 0$ for suitable j. Hence we may assume that either $\beta_1=1$ or that $c_{13}=0$, $c_{23}=\gamma_8u_8$. If $\delta_1\neq 0$, apply $\exp(2(0, u_1, 0, 0))$, leaving $u_1(1, 2)$

fixed and making $c_{13}\neq 0$. Consequently we may assume either $\beta_1=1$ or $c=c_{12}(1, 2)+\gamma_8 u_8(2, 3)$, and $\alpha_5=0$ by earlier observations. If $\alpha_i\neq 0$, $i\in D(1)$, $i\neq 1$, follow B by suitable $\exp(2(0, 0, u_j, 0))$, $j\in D(1)$, $j\neq 8$, leaving $u_1(1, 2)$ fixed and making $c_{13}\neq 0$, so we may assume $c_{12}=\sum_{i=1}^4 \alpha_i u_i$. If $\gamma_8\neq 0$, apply $\exp(2(u_6, 0, 0, 0))$, leaving $u_1(1, 2)$ fixed and making $c_{13}\neq 0$. Thus either $\beta_1=1$ or $c=\sum_{i=1}^4 \alpha_i u_i(1, 2)$ may be assumed. In the second case, if $\alpha_i\neq 0$, $i\neq 1$, follow by $\exp(2(u_{i+4}, 0, 0, 0))$, leaving $u_1(1, 2)$ fixed and making $\delta_1\neq 0$. Thus we may assume $\beta_1=1$ unless $u_1(1, 3)B=\alpha_1 u_1(1, 2)=\alpha_1 u_1(1, 2)B$; the latter is impossible since B is an automorphism, and we may therefore assume

$$\beta_1 = 1,\ c = \delta_1 w_1 + \sum_{i\neq 5} \alpha_i u_i(1, 2) + u_1(1, 3) + \sum_{i\in D(1), i\neq 1} \beta_i u_i(1, 3)$$
$$+ \sum_{i\in D(1)} \gamma_i u_i(2, 3).$$

Now we show that we may assume $u_1(1, 3)B=u_1(1, 3)$. First operate with $\exp(2\lambda_i(u_i, 0, 0, 0))$, $i=2, 3, 4$; these leave $u_1(1, 2)$ fixed, the coefficient 1 of u_1 in c_{13} is left fixed, and for $j\in D(1)$, $j\neq 1$, and $i=2, 3, 4$, $\bar{u}_i u_j=0$, $-u_8$, or $2u_8$. Also $\bar{u}_i u_1$ is a nonzero scalar multiple of u_7, u_6, u_1, respectively. Hence with suitable λ_i we make c_{23} into $\gamma_8 u_8$. Next operate with $\exp(2\lambda(0, 0, u_8, 0))$; $u_1(1, 2)$ is fixed, the coefficient of u_1 in c_{13} remains 1, and c_{23} retains its form. The coefficient α_1 of u_1 in c_{12} is replaced by $\alpha_1-2\lambda$, so becomes zero for $\lambda=2^{-1}\alpha_1$. We may therefore assume that $\alpha_1=0$. Now apply $\exp(2\lambda(0, 0, 0, T))$ for $T=E_{74}-E_{83}$, $E_{64}-E_{82}$, $E_{63}-E_{72}$. Again $u_1(1, 2)$ is fixed, as is the coefficient 1 of u_1 in c_{13}, the coefficient 0 of u_1 in c_{12}, and, from the values of T^ϕ, $c_{23}=\gamma_8 u_8$ is unchanged. The values of T^ψ show that for suitable λ we can make $\beta_6=\beta_7=\beta_8=0$ and thus assume $c=\delta_1 w_1+\sum_{i\neq 1,5}\alpha_i u_i(1, 2)+u_1(1, 3)+\gamma_8 u_8(2, 3)$. The entry in the (1, 2)-position of $c\cdot c$ is $4\gamma_8 u_1 u_4=4\gamma_8 u_1$. But $c\cdot c=0$, since $u_1(1, 3)\cdot u_1(1, 3)=0$; hence $\gamma_8=0$. Then the entry in the (1, 3)-position of $c\cdot c$ is $2\delta_1 u_1$, so $\delta_1=0$. The entry in the (2, 3)-position of $c\cdot c$ is now $2\bar{c}_{12}u_1=\alpha_2 u_7-\alpha_3 u_6+4\alpha_4 u_1$, so $\alpha_2=\alpha_3=\alpha_4=0$, and $c=(\alpha_6 u_6+\alpha_7 u_7+\alpha_8 u_8)(1, 2)+u_1(1, 3)=u_1(1, 3)B$. Next we observe that $u_1(1, 2)\cdot u_4(2, 3)=u_1(1, 3)$, so that $u_1(1, 2)\cdot u_4(2, 3)B=u_1(1, 3)B=c$. However, the entry in the (1, 2)-position of $u_1(1, 2)\cdot d$ for any $d\in J$ is a scalar multiple of u_1. Hence $\alpha_6=\alpha_7=\alpha_8=0$, and $u_1(1, 3)B=u_1(1, 3)$. Thus we may assume both $u_1(1, 2)$ and $u_1(1, 3)$ fixed under B.

Now if $c=u_4(2, 3)B$ as above, we have $u_1(1, 2)\cdot c=u_1(1, 3)$, from the last paragraph, and $u_1(1, 3)\cdot c=0$ by $u_1(1, 3)\cdot u_4(2, 3)=0$. It follows that $\alpha_5=0=\delta_2=\beta_2=\beta_3=\beta_4=\beta_5=\gamma_2=\gamma_3=\gamma_5=\alpha_2=\alpha_3=\alpha_4=\alpha_8=\delta_1-\delta_2=\gamma_4-1$. Thus $c=\sum_{i\in D(1)}\alpha_i u_i(1, 2)+\sum_{i\in D(1)}\beta_i u_i(1, 3)+\sum_{i\in D(1), i\neq 8}\gamma_i u_i(2, 3)+u_4(2, 3)$. Now apply $\exp(2\beta_1(u_1, 0, 0, 0))$, leaving $u_1(1, 2)$ and $u_1(1, 3)$ fixed, and replacing β_1 by 0. Thus we may assume $\beta_1=0$. Next apply $\exp(2\alpha_i(0, u_i, 0, 0))$, $i=6, 7, 8$, leaving $u_1(1, 2)$, $u_1(1, 3)$, c_{13} unchanged, and replacing α_i by 0, $i=6, 7, 8$. Thus we assume $c=\alpha_1 u_1(1, 2)+\sum_{i\in D(1), i\neq 1}\beta_i u_i(1, 3)+u_4(2, 3)$

$+\sum_{i\in D(1),i\neq 8}\gamma_i u_i(2, 3)$. Next operate with $\exp(2\lambda(0, 0, 0, T))$, $T=E_{54}-E_{81}$, $E_{53}-E_{71}$, $E_{52}-E_{61}$; $u_1(1, 2)$ and c_{12} are unchanged, and the values of T^{ψ} show that $u_1(1, 3)$ and c_{13} are unchanged. Those of T^{ϕ} show that suitable values of λ make all γ_i, $i = 1, 6, 7$, into 0. Thus we may assume $c = \alpha_1 u_1(1, 2) + \sum_{i\in D(1),i\neq 1}\beta_i u_i(1, 3)+u_4(2, 3)$. From $c\cdot c=0$ we now have $c_{13}\bar{u}_4=0=c_{13}u_8=2c_{13}$, so $c_{13}=0$; also $\alpha_1 u_1 u_4=0=\alpha_1 u_1$; hence $\alpha_1=0$, and $c=u_4(2, 3)$. Thus the following may be assumed fixed under B: $u_1(1, 2)$, $u_1(1, 3)$, $u_4(2, 3)$.

As above, we now have $u_1(1, 2)\cdot u_1(2, 3)B=0=u_1(1, 3)\cdot u_1(2, 3)B=u_4(2, 3)\cdot u_1(2, 3)B$. It follows that if $u_1(2, 3)B=c$ as above, then

$$c = (\alpha_6 u_6 + \alpha_7 u_7 + \alpha_8 u_8)(1, 2) + \beta_1 u_1(1, 3) + (\gamma_1 u_1 + \gamma_6 u_6 + \gamma_7 u_7)(2, 3).$$

From $u_1(2, 3)B= -u_1(1, 2)\cdot u_4(1, 3)B$, we have $\alpha_6=\alpha_7=\alpha_8=0$. Moreover, not all of $\gamma_1, \gamma_6, \gamma_7$ can be zero, since $u_1(2, 3)B\neq\beta_1 u_1(1, 3)B=\beta_1 u_1(1, 3)$. Now consider $\exp(2\lambda(0, 0, 0, T))$, where T is chosen from among $E_{42}-E_{68}$, $E_{43}-E_{78}$, $E_{34}-E_{87}$, $E_{24}-E_{86}$. Then $T^{\psi}=T$ and T^{ϕ} is among $(E_{17}-E_{35})/2$, $-(E_{16}-E_{25})/2$, $2(E_{52}-E_{61})$, $-2(E_{53}-E_{71})$, respectively. Thus $u_1(1, 2)$, $u_1(1, 3)$, $u_4(2, 3)$ are fixed, and by suitable choice of λ we may use one of the first two types to make either $\gamma_6\neq 0$ or $\gamma_7\neq 0$. Then suitable choice of λ and one of the last two types makes $\gamma_1=1$. The first two types may then be used to make $\gamma_6=\gamma_7=0$, with $\gamma_1=1$. We may therefore assume $u_1(2, 3)B=\beta_1 u_1(1, 3)+u_1(2, 3)$. Finally, an application of an appropriate $\exp(2\lambda(u_8, 0, 0, 0))$ makes β_1 into 0 and leaves all other properties of B fixed. We may therefore assume $u_1(j, k)$, $1\leqq j<k\leqq 3$, and $u_4(2, 3)$ fixed under B.

From $u_1(1, 2)\cdot u_4(1, 3)B= -u_1(2, 3)$, $u_1(2, 3)\cdot u_4(1, 3)B=0=u_1(2, 3)\cdot u_4(1, 3)B=u_4(2, 3)\cdot u_4(1, 3)B$, it follows as above that $u_4(1, 3)B=(\alpha_6 u_6+\alpha_7 u_7)(1, 2)+(\beta_1 u_1+u_4)(1, 3)+(\gamma_1 u_1+\gamma_6 u_6+\gamma_7 u_7)(2, 3)$. Now operate with $\exp(2\lambda(0, 0, u_i, 0))$, $i=6, 7$. The $u_1(j, k)$ and $u_4(2, 3)$ are fixed and for suitable values of λ the coefficients α_6, α_7 are replaced by 0. Thus we may assume $\alpha_6=\alpha_7=0$. Next operate with $\exp(2\lambda(0, 0, 0, T))$, $T=E_{41}-E_{58}$; $T^{\psi}=T$ and $T^{\phi}= -(E_{27}-E_{36})/2$, so that all $u_1(j, k)$ and $u_4(2, 3)$ are fixed, and $u_4(1, 3)B$ is changed only by the replacement of β_1 by $\beta_1+\lambda$. With $\lambda= -\beta_1$, we make β_1 into 0, and may assume $u_4(1, 3)B=u_4(1, 3)+(\gamma_1 u_1+\gamma_6 u_6+\gamma_7 u_7)(2, 3)$. Since this element must have square zero, $\gamma_6=\gamma_7=0$; from $u_4(1, 3)B=u_4(1, 2)B\cdot u_4(2, 3)$, it follows that $\gamma_1=0$, and $u_4(1, 3)$ may also be assumed fixed under B.

Next consider $u_4(1, 2)B$. From $u_1(1, 2)\cdot u_4(1, 2)B=0=u_4(1, 2)B\cdot u_1(2, 3)=u_4(1, 2)B\cdot u_4(1, 3)$, and from $u_4(1, 2)B\cdot u_1(1, 3)=u_1(2, 3)$, $u_4(1, 2)B\cdot u_4(2, 3)=u_4(1, 3)$, we see as before that $u_4(1, 2)B=(u_4+\alpha_6 u_6+\alpha_7 u_7)(1, 2)+\beta_1 u_1(1, 3)+\gamma_1 u_1(2, 3)$, and from $u_4(1, 2)B\cdot u_4(1, 2)B = 0$, that $\beta_1 = 0$. Now $\exp(2\lambda(0, u_1, 0, 0))$ leaves fixed all $u_1(j, k)$, as well as $u_4(1, 3)$ and $u_4(2, 3)$; in $u_4(1, 2)B$ it replaces γ_1 by $\gamma_1+\lambda$, so that $\lambda= -\gamma_1$ makes γ_1 into 0, and enables us to assume $u_4(1, 2)B=(u_4+\alpha_6 u_6+\alpha_7 u_7)(1, 2)$. Now apply $\exp(2\lambda(0, 0, 0, T))$, $T=E_{28}-E_{46}$, $E_{38}-E_{47}$; then all our fixed elements remain fixed, the form of

$u_4(1, 2)B$ is unchanged, and we can make $\alpha_6=\alpha_7=0$ by suitable choices of λ. Thus we may assume all $u_1(j, k)$ and all $u_4(j, k)$ fixed under B.

From $u_1(1, 2)\cdot u_8(1, 2)=0=u_8(1, 2)\cdot u_1(1, 3)=u_8(1, 2)\cdot u_1(2, 3)-2u_1(1, 3)$, we have $u_8(1, 2)B=(\alpha_6u_6+\alpha_7u_7+u_8)(1, 2)+\beta_1u_1(1, 3)+(\gamma_1u_1+\gamma_6u_6+\gamma_7u_7)(2, 3)$. Now apply $\exp(2\lambda(0, 0, 0, T))$, $T=E_{34}-E_{87}$, $E_{24}-E_{86}$; all $u_1(j, k)$ and $u_4(j, k)$ remain fixed, and suitable choices of λ make α_6 and α_7 into 0. Next, applying $\exp(2\lambda(0, 0, u_1, 0))$ does not change any $u_1(j, k)$, $u_4(j, k)$, α_6 or α_7, and for $\lambda=-\beta_1/2$ replaces β_1 by 0. We may thus assume $u_8(1, 2)B=u_8(1, 2)+(\gamma_1u_1+\gamma_6u_6+\gamma_7u_7)(2, 3)$. The fact that $u_8(1, 2)$ has square zero now implies $\gamma_1=0$, and $u_4(2, 3)=u_8(1, 2)\cdot u_4(1, 3)$ yields $\gamma_6=\gamma_7=0$. Hence we may assume $u_8(1, 2)B=u_8(1, 2)$, giving another basis element fixed under B. From $u_4(1, 2)\cdot u_8(1, 2)=2(w_1+2w_2+2I)/3$, we see that w_1+2w_2+2I is fixed under B, hence that w_1+2w_2 is fixed under B.

Now we find as above that $u_8(2, 3)B=(\alpha_6u_6+\alpha_7u_7)(1, 2)+(\beta_6u_6+\beta_7u_7)(1, 3)+(\gamma_1u_1+u_8)(2, 3)$. Applying $\exp(2\lambda(u_i, 0, 0, 0))$, $i=6, 7$, leaves $u_1(j, k)$, $u_4(j, k)$ and $u_8(1, 2)$ fixed and for suitable values of λ gives $\beta_6=\beta_7=0$. Thus we may assume $\beta_6=\beta_7=0$. Applying $\exp(2\lambda(0, 0, 0, T))$ for $T=E_{27}-E_{36}$ leaves these coefficients unchanged, as well as $u_1(j, k)$, $u_4(j, k)$ and $u_8(1, 2)$, and for suitable λ makes γ_1 into 0. Thus we may assume $u_8(2, 3)B=(\alpha_6u_6+\alpha_7u_7)(1, 2)+u_8(2, 3)$. From the fact that $u_8(2, 3)$ has square 0, $\alpha_6=\alpha_7=0$, and $u_8(2, 3)$ may also be assumed fixed. Then $u_4(2, 3)\cdot u_8(2, 3)$ is fixed, so $2w_1+w_2$ is fixed as above. It follows that w_1 and w_2 are both fixed. From $u_8(1, 2)\cdot u_8(2, 3)=2u_8(1, 3)$ it follows that $u_8(1, 3)$ is fixed. Thus w_1, w_2, and $u_i(j, k)$, $i=1, 4, 8$, may be assumed fixed under B.

Now as before we find that $u_5(1, 2)B=(\alpha_1u_1+u_5+\alpha_6u_6+\alpha_7u_7)(1, 2)$, and from $u_5(1, 2)\cdot u_5(1, 2)=0$, that $\alpha_1=0$. Now apply $\exp(2\lambda(0, 0, 0, T))$ for $T=E_{21}-E_{56}$, $E_{31}-E_{57}$ to make $\alpha_6=\alpha_7=0$, while leaving fixed all $u_i(j, k)$, $i = 1, 4, 8$. Thus we may assume $u_5(1, 2)$ fixed under B. Since $2u_5(1, 3)=u_5(1, 2)\cdot u_8(2, 3)$, $-2u_5(2, 3)=u_5(1, 2)\cdot u_8(1, 3)$, all $u_5(j, k)$ are fixed, and all of w_1, w_2, $u_i(j, k)$, $i=1, 4, 5, 8$ may be assumed fixed.

As before, we now have $u_2(1,2)B=(\alpha_2u_2+\alpha_3u_3)(1,2)$. Now apply $\exp(2\lambda(0, 0, 0, T))$ for $T=E_{23}-E_{76}$, $E_{32}-E_{67}$ to obtain first $\alpha_3\neq 0$, then $\alpha_2=1$, then $\alpha_3=0$, as in similar situations previously. All elements formerly fixed remain fixed, and $u_2(1, 2)$ is now fixed. Thus we may assume $u_2(1, 2)B=u_2(1, 2)$. From $u_2(1, 2)\cdot u_4(2, 3)=u_2(1, 3)$, $u_2(1, 2)\cdot u_4(1, 3)=-u_2(2, 3)$, $u_2(1, 2)\cdot u_1(2, 3)=-u_7(1, 3)/2$, $u_2(1, 2)\cdot u_1(1, 3)=u_7(2, 3)/2$, $u_2(1, 3)\cdot u_1(2, 3)=u_7(1, 2)/2$, we see that all $u_2(j, k)$, $u_7(j, k)$ are fixed, therefore that all $u_i(j, k)$, $i\neq 3, 6$, may be assumed fixed.

It now follows as above that $u_3(1, 2)B=(\alpha_2u_2+u_3)(1, 2)$. Then we apply $\exp(-2\alpha_2(0, 0, 0, E_{32}-E_{67}))$, without changing any of our fixed vectors, to obtain $\alpha_2=0$, i.e., $u_3(1, 2)B=u_3(1, 2)$. As in the case of $u_2(1, 2)$ it follows that all $u_3(j, k)$, $u_6(j, k)$ may be assumed fixed. Thus B has been reduced to the identity, we have shown that $A\mathfrak{G}A^{-1}=\mathfrak{A}(\mathfrak{J})$, and therefore $G'=\mathfrak{g}(\mathfrak{L})=\mathfrak{A}(\mathfrak{L})\cong\mathfrak{A}(\mathfrak{J})$. In summary, we have the

THEOREM 3. *If $\mathfrak{L}$ is the Lie algebra of derivations of the split exceptional Jordan algebra $\mathfrak{J}$ over a field of characteristic different from* 2, 3, 5, 7, 11, 13, *then every automorphism of $\mathfrak{L}$ is an invariant automorphism*; *in fact, the automorphism group $\mathfrak{A}(\mathfrak{L})$ is generated by all* $\exp(\lambda\,\mathrm{ad}(X))$, *where X runs through a set of root-vectors relative to a fixed standard Cartan subalgebra. The identification of the simple group G' of Chevalley, formed over the field $\mathfrak{F}$ from a complex Lie algebra of type F_4, with such a subgroup of $\mathfrak{A}(\mathfrak{L})$ shows that* $G'=\mathfrak{s}(\mathfrak{L})=\mathfrak{A}(\mathfrak{L})\cong\mathfrak{A}(\mathfrak{J})$, *the group of automorphisms of* $\mathfrak{J}$.

5. **Norm-skew transformations of $\mathfrak{J}$.** Let $\mathfrak{J}$ be the split exceptional Jordan algebra over the field $\mathfrak{F}$ of characteristic $\neq 2, 3$. If $a=\mathrm{diag}\{\alpha_{11}, \alpha_{22}, \alpha_{33}\}+a_{12}(1, 2)+a_{13}(1, 3)+a_{23}(2, 3)\in\mathfrak{J}$, set $\mathrm{Tr}(a)=\alpha_{11}+\alpha_{22}+\alpha_{33}$. Then one readily verifies that $(a, b)=\mathrm{Tr}(a\cdot b)$ is a symmetric bilinear form on $\mathfrak{J}$, and is *associative* (*invariant*) in the sense that $(a\cdot c, b)=(a, c\cdot b)$ for all $a, b, c\in\mathfrak{J}$. The form is nondegenerate; for if $b=\mathrm{diag}\{\beta_{11}, \beta_{22}, \beta_{33}\}+b_{12}(1, 2)+b_{13}(1, 3)+b_{23}(2, 3)$, then

$$(a, b) = 2(\alpha_{11}\beta_{11} + \alpha_{22}\beta_{22} + \alpha_{33}\beta_{33}) + 4[(a_{12}, b_{12}) + (a_{13}, b_{13}) + (a_{23}, b_{23})].$$

Thus if $(a, b)=0$ for all b, we must have $\alpha_{11}\beta_{11}+\alpha_{22}\beta_{22}+\alpha_{33}\beta_{33}=0$ for all $\beta_{11}, \beta_{22}, \beta_{33}\in\mathfrak{F}$, hence $\alpha_{11}=\alpha_{22}=\alpha_{33}=0$; also $(a_{ij}, b_{ij})=0$ for all $b_{ij}\in\mathfrak{C}$, hence all $a_{ij}=0$ by the nondegeneracy of the form on $\mathfrak{C}$, and $a=0$.

We have seen in §1 that $\mathrm{Tr}(aD)=0$ for all $a\in\mathfrak{J}$ and all derivations D of $\mathfrak{J}$. Thus

$$0 = \mathrm{Tr}((a\cdot b)D) = \mathrm{Tr}((a\cdot bD) + (aD\cdot b)) = (a, bD) + (aD, b) \tag{29}$$

for all $a, b\in\mathfrak{J}$, and *every derivation of $\mathfrak{J}$ is skew with respect to* (a, b). From the associativity of (a, b), it follows that the trilinear form $\tau(a, b, c)=(a\cdot b, c)$ is a symmetric function of its variables; for $\tau(a, b, c)=\tau(b, a, c)$ by commutativity in $\mathfrak{J}$, and $\tau(a, b, c)=\tau(c, b, a)$ by associativity and the symmetry of (a, b). The result follows. Following Freudenthal [4], we define a modified trilinear form by

$$\begin{aligned}(a, b, c) = \tau(a, b, c) - \mathrm{Tr}(a)(b, c) - \mathrm{Tr}(b)(a, c) - \mathrm{Tr}(c)(a, b)\\ + 2\mathrm{Tr}(a)\mathrm{Tr}(b)\mathrm{Tr}(c).\end{aligned} \tag{30}$$

This form is evidently symmetric. If $a\in\mathfrak{J}$ and if D is a derivation of $\mathfrak{J}$, then $(a, a, aD) = \tau(a, a, aD) - \mathrm{Tr}(a)((aD, a) + (a, aD)) - \mathrm{Tr}(aD)(a, a) +2\,\mathrm{Tr}(aD)\mathrm{Tr}(a)^2=\tau(a, a, aD)$ by (29) and the fact that $\mathrm{Tr}(aD)=0$. Now $\tau(a, a, aD)=\mathrm{Tr}(a\cdot(a\cdot aD))=\mathrm{Tr}(a\cdot(a\cdot a)D)-\mathrm{Tr}(a\cdot(aD\cdot a))=\mathrm{Tr}((a\cdot(a\cdot a))D) -\mathrm{Tr}(aD\cdot(a\cdot a))-\mathrm{Tr}(a\cdot(aD\cdot a))=\mathrm{Tr}((a\cdot(a\cdot a))D)-\tau(aD, a, a)-\tau(a, aD, a)$, or by the symmetry of τ, $3\tau(a, a, aD)=\mathrm{Tr}((a\cdot(a\cdot a))D)=0$ by §1, and thus $(a, a, aD)=0$. Polarization yields

$$(aD, b, c) + (a, bD, c) + (a, b, cD) = 0 \tag{31}$$

for all $a, b, c\in\mathfrak{J}$ and all derivations D of $\mathfrak{J}$.

Conversely, if D is any linear transformation of $\mathfrak{J}$ which is skew with respect to both the bilinear and trilinear forms, then D is a derivation of $\mathfrak{J}$. For, first of all, $\mathrm{Tr}(aD)=0$ for all $a\in\mathfrak{J}$; this follows since $(I, ID)=0 =\mathrm{Tr}(I\cdot ID)=2\,\mathrm{Tr}(ID)$ gives $\mathrm{Tr}(ID)=0$, and then $(a, I, ID)+(a, ID, I) +(aD, I, I)=0$ yields $0=2\,\mathrm{Tr}(a\cdot(I\cdot ID))+\mathrm{Tr}(aD\cdot(I\cdot I))-2\,\mathrm{Tr}(a)\mathrm{Tr}(I\cdot ID) -\mathrm{Tr}(aD)\mathrm{Tr}(I\cdot I)-2\,\mathrm{Tr}(ID)\mathrm{Tr}(a\cdot I)-2\,\mathrm{Tr}(I)\mathrm{Tr}(a\cdot ID)-2\,\mathrm{Tr}(I)\mathrm{Tr}(I\cdot aD) +4\,\mathrm{Tr}(a)\mathrm{Tr}(ID)\mathrm{Tr}(I)+2\,\mathrm{Tr}(aD)\mathrm{Tr}(I)^2=-2(a, ID)+4\,\mathrm{Tr}(aD)=-2(a, ID) - 4\,\mathrm{Tr}(aD)+8\,\mathrm{Tr}(aD) = -2(a, ID)-2\,\mathrm{Tr}(aD\cdot I)+8\,\mathrm{Tr}(aD) =-2((a, ID)+(aD, I))+8\,\mathrm{Tr}(aD)=8\,\mathrm{Tr}(aD)$. Thus $\mathrm{Tr}(aD)=0$. Then for a, b, c arbitrary in $\mathfrak{J}$, we have $((a\cdot b)D-a\cdot bD-aD\cdot b, c)=-(a\cdot b, cD) -(a\cdot bD, c)-(aD\cdot b, c)=-\tau(a, b, cD)-\tau(a, bD, c)-\tau(aD, b, c)=-(a, b, cD) +\mathrm{Tr}(a)(b, cD)+\mathrm{Tr}(b)(a, cD)-(a, bD, c)+\mathrm{Tr}(a)(bD, c)+\mathrm{Tr}(c)(a, bD) -(aD, b, c)+\mathrm{Tr}(b)(aD, c)+\mathrm{Tr}(c)(aD, b)$, the remaining terms vanishing because $\mathrm{Tr}(aD)=\mathrm{Tr}(bD)=\mathrm{Tr}(cD)=0$. By (31), this is $\mathrm{Tr}(a)((b, cD)+(bD, c)) +\mathrm{Tr}(b)((a, cD)+(aD, c))+\mathrm{Tr}(c)((a, bD)+(aD, b))$; by the skewness of D with respect to (x, y), this quantity vanishes. From the nondegeneracy of (x, y), it follows that $(a\cdot b)D-a\cdot bD-aD\cdot b=0$, thus that D is a derivation of $\mathfrak{J}$.

Next let $d\in\mathfrak{J}$, $\mathrm{Tr}(d)=0$, and consider $(a, a, a\cdot d)$. A lengthy but straightforward computation in $\mathfrak{C}$ shows this to be zero. Then for all $a, b, c\in\mathfrak{J}$, $(a\cdot d, b, c) +(a, b\cdot d, c)+(a, b, c\cdot d)=0$, and the right multiplication R_d satisfies (31). In place of (29), we have

$$(32)\qquad (aR_d, b) = (a\cdot d, b) = (a, d\cdot b) = (a, b\cdot d) = (a, bR_d),$$

or R_d is *symmetric* with respect to the bilinear form.

Now let $\mathfrak{L}$ be the space of linear transformations of $\mathfrak{J}$ satisfying (31). Then $\mathfrak{L}$ is a Lie algebra, and if the characteristic of $\mathfrak{F}$ is a prime $p(>3)$, then $T^p\in\mathfrak{L}$ for all $T\in\mathfrak{L}$ [10], and so $\mathfrak{L}$ is a restricted Lie algebra under pth iterates in this case.

If R and S are symmetric linear transformations of $\mathfrak{J}$ with respect to (x, y), then $[RS]$ is skew. In particular if $a, b\in\mathfrak{J}'$, then $[R_aR_b]$ is skew with respect to both the bilinear and trilinear forms, hence is a derivation of $\mathfrak{J}$ by the above. Moreover, if $a\in\mathfrak{J}$ and if D is a derivation of $\mathfrak{J}$, we have $aD\in\mathfrak{J}'$, and $[R_aD]=R_{aD}$. Since (x, y) is nondegenerate on $\mathfrak{J}$, the only transformation which is both skew and symmetric with respect to this form is 0. If $\mathfrak{D}$ denotes the derivation algebra of $\mathfrak{J}$, we thus have $\mathfrak{D}\cap\mathfrak{R}_{\mathfrak{J}}=(0)$, where the meaning of $\mathfrak{R}_{\mathfrak{J}}$ is clear. Therefore $\mathfrak{R}_{\mathfrak{J}'}+\mathfrak{D}=\mathfrak{R}_{\mathfrak{J}'}\oplus\mathfrak{D}$, and is a Lie-subalgebra of $\mathfrak{L}$ by the remarks above. We prove that $\mathfrak{L}=\mathfrak{R}_{\mathfrak{J}'}\oplus\mathfrak{D}$ (observation of some of our later remarks will show that $\mathfrak{L}$ is generated by $\mathfrak{R}_{\mathfrak{J}'}$). $\mathfrak{L}$ will be called the Lie algebra of *norm-skew* transformations of $\mathfrak{J}$.

If $T\in\mathfrak{L}$, then $(I, I, IT) = 0 = \mathrm{Tr}((I\cdot I)\cdot IT) - 2\,\mathrm{Tr}(I)\mathrm{Tr}(I\cdot IT) -\mathrm{Tr}(IT)\mathrm{Tr}(I\cdot I)+2\,\mathrm{Tr}(IT)\mathrm{Tr}(I)^2=4\,\mathrm{Tr}(IT)$. Thus $IT\in\mathfrak{J}'$. Now consider $D=T-R_{IT}/2\in\mathfrak{L}$. We show $D\in\mathfrak{D}$. If $a\in\mathfrak{J}$, we have $(aD, a, I)+(a, aD, I)$

$+(a, a, ID)=0$, or $2(a, aD, I)+(a, a, IT)-(a, a, I\cdot IT)/2=0=2(a, aD, I)$, or $(a, aD, I)=0$. But $(a, aD, I)=-(a, aD)+2\,\mathrm{Tr}(aD)\mathrm{Tr}(a)$. Thus $(a, aD)=2\,\mathrm{Tr}(aD)\mathrm{Tr}(a)$. From $ID=0$, we have further that $0=(aD, I, I)+(a, ID, I)+(a, I, ID)=(aD, I, I)=4\,\mathrm{Tr}(aD)$, so that $\mathrm{Tr}(aD)=0$, and $(a, aD)=0$ by the above. Thus D is skew with respect to (x, y), so is a derivation of $\mathfrak{J}$. We thus have

$$T = D + R_{(I/2)T} \in \mathfrak{D} \oplus \mathfrak{R}_{\mathfrak{J}'}, \tag{33}$$

and $\mathfrak{L}=\mathfrak{D}\oplus\mathfrak{R}_{\mathfrak{J}'}$, as asserted. It follows that $\mathfrak{L}$ has dimension 78 (since $I\in\mathfrak{J}$, $R_a=R_b$ implies $a=b$). We now show that $\mathfrak{L}$ is the simple Lie algebra of classical type E_6 over $\mathfrak{F}$.

In notations already established in §1, the linear span of the derivations $(0, 0, 0, H_i)$, $1\leqq i\leqq 4$, is a Cartan subalgebra of $\mathfrak{D}$. In $\mathfrak{L}$, let $\mathfrak{H}$ be the subspace spanned by these four derivations and by the multiplications $h_5=R_{w_1}$, $h_6=R_{w_2}$. We show that $\mathfrak{H}$ is an abelian Cartan subalgebra relative to which (i)–(v) of [8] are satisfied, and we shall display a fundamental system of roots relative to $\mathfrak{H}$ which is of type E_6. The simplicity of $\mathfrak{L}$ will then follow immediately.

From §1, we see that each w_j, $j=1, 2$, is annihilated by each of the derivations $h_i=2(0, 0, 0, H_i)$, $1\leqq i\leqq 4$, hence that $[h_jh_i]=0$, $j=5, 6$, $1\leqq i\leqq 4$, by $[R_aD]=R_{aD}$. Now we know that $[h_5h_6]$ is a derivation $D=(b_{12}, b_{13}, b_{23}, T)$, where $w_1D=b_{12}(1, 2)+b_{13}(1, 3)/2-b_{23}(2, 3)/2$, $c(1, 2)D=\mathrm{diag}\{\gamma_{11}, \gamma_{22}, \gamma_{33}\}+(cT)(1, 2)/2+c_{13}(1, 3)+c_{23}(2, 3)$, for some γ_{ii}, c_{jk}. One readily calculates that $w_1[h_5h_6]=0=c(1, 2)[h_5h_6]$, so that $b_{12}=b_{13}=b_{23}=0$ and $T=0$; i.e., $D=0$, and $\mathfrak{H}$ is a 6-dimensional commutative subalgebra. We also note:

$$\begin{aligned} &c(1, 2)h_5 = 0;\ c(1, 2)h_6 = c(1, 2);\ w_1h_5 = \mathrm{diag}\{2, 2, 0\};\\ &w_2h_5 = \mathrm{diag}\{0, -2, 0\} = w_1h_6;\ w_2h_6 = \mathrm{diag}\{0, 2, 2\};\\ &c(1, 3)h_5 = c(1,3) = -\,c(1,3)h_6;\ c(2,3)h_5 = -\,c(2,3);\ c(2,3)h_6 = 0. \end{aligned} \tag{34}$$

Moreover, if D is of the form $(0, 0, 0, T)$, we have $[h_iD]=0$, $i=5, 6$. Then, as T runs through the 24 transformations of $\mathfrak{C}$: $E_{ij}-E_{j+4,i+4}$, $i\neq j$; $E_{i+4,j}-E_{j+4,i}$, $i<j$; $E_{i,j+4}-E_{j,i+4}$, $i<j$; $1\leqq i, j\leqq 4$, as in §1, we have linear functions $\gamma_1, \cdots, \gamma_{12}$ on $\mathfrak{H}$ with $\gamma_i(h_5)=0=\gamma_i(h_6)$, $1\leqq i\leqq 12$, and where $\gamma_i(h_j)$ has one of the forms $\delta_{mj}-\delta_{nj}$, $-\delta_{mj}-\delta_{nj}$ for fixed m, n, $1\leqq m<n\leqq 4$, and all j, $1\leqq j\leqq 4$, such that $[(0, 0, 0, T), h]=\pm\gamma(h)(0, 0, 0, T)$ for all T above, where γ runs through $\gamma_1, \cdots, \gamma_{12}$, and where each $(0, 0, 0, T)$ corresponds to a different function from $\pm\gamma_1, \cdots, \pm\gamma_{12}$.

For $a\in\mathfrak{C}$, we let $a(1)=R_{a(1,2)}$, $a(2)=R_{a(1,3)}$, $a(3)=R_{a(2,3)}\in\mathfrak{L}$. Then we have

$$\begin{aligned} &[u_i(1) \pm 2(u_i, 0, 0, 0), h_i] = u_i(1) \pm 2(u_i, 0, 0, 0),\\ &[u_{i+4}(1) \pm 2(u_{i+4}, 0, 0, 0), h_i] = -\,(u_{i+4}(1) \pm 2(u_{i+4}, 0, 0, 0)),\\ &[u_i(1) \pm 2(u_i, 0, 0, 0), h_j] = 0 = [u_{i+4}(1) \pm 2(u_{i+4}, 0, 0, 0), h_j], \end{aligned} \tag{35}$$

$1\leqq i, j\leqq 4$, $i\neq j$. These follow readily from §1 and $[R_aD]=R_{aD}$.

We also have in general $[b(1)\pm 2(b, 0, 0, 0), h_5]=[R_{b(1,2)}h_5]\mp 2R_{w_1(b,0,0,0)}$. Now $w_1[b(1), h_5]=-\mathrm{diag}\{2, 2, 0\}b(1)=-4b(1, 2)=-2w_1(2(b, 0, 0, 0))$, while for $c\in C$, $(c(1, 2)[b(1), h_5])_{12}=0$ is readily checked. Thus the derivation $[b(1), h_5]$ is $-4(b, 0, 0, 0)$. Also, $w_1(b, 0, 0, 0)=b(1, 2)$, so $R_{w1(b,0,0,0)}=b(1)$, yielding, together with analogous computations for h_6,

$$(36)\qquad \begin{aligned}[][b(1) \pm 2(b, 0, 0, 0), h_5] &= \mp 2(b(1) \pm 2(b, 0, 0, 0)),\\ [b(1) \pm 2(b, 0, 0, 0), h_6] &= \pm (b(1) \pm 2(b, 0, 0, 0)).\end{aligned}$$

Thus as b runs through the basis $u_1, \cdots, u_8$ of $\mathfrak{C}$, we have 16 distinct linear functions γ on H with $[b(1)\pm 2(b, 0, 0, 0), h]=\gamma(h)(b(1)\pm 2(b, 0, 0, 0))$ for all $h\in\mathfrak{H}$. These consist of eight functions $\gamma_{13}, \cdots, \gamma_{20}$ and their negatives, with $\gamma_j(h_5)=-2$, $\gamma_j(h_6)=1$, $13\leqq j\leqq 16$, and with $\gamma_j(h_5)=2$, $\gamma_j(h_6)=-1$, $17\leqq j\leqq 20$. For $13\leqq j\leqq 16$, we have $\gamma_j(h_i)=\delta_{j-12,i}$, $1\leqq i\leqq 4$, and for $17\leqq j\leqq 20$, $\gamma_j(h_i)=\delta_{j-16,i}$, $1\leqq i\leqq 4$. For $1\leqq i\leqq 4$, $u_i(1)+2(u_i, 0, 0, 0)$ belongs to γ_{i+12}, $u_{i+4}(1)-2(u_{i+4}, 0, 0, 0)$ to $-\gamma_{i+12}$, $u_i(1)-2(u_i, 0, 0, 0)$ to γ_{i+16}, $u_{i+4}(1)+2(u_{i+4}, 0, 0, 0)$ to $-\gamma_{i+16}$.

Similarly, we have eight linear functions $\gamma_{21}, \cdots, \gamma_{28}$ on $\mathfrak{H}$, with $\gamma_j(h_i)=-1/2+\delta_{j-20,i}$, $21\leqq j\leqq 24$, $1\leqq i\leqq 4$, $\gamma_j(h_5)=-1=\gamma_j(h_6)$, $21\leqq j\leqq 24$, and with $\gamma_j(h_5)=1=\gamma_j(h_6)$, $25\leqq j\leqq 28$, $\gamma_j(h_i)=-1/2+\delta_{j-24,i}$, $25\leqq j\leqq 28$, $1\leqq i\leqq 4$. For $1\leqq i\leqq 4$, $u_i(2)+2(0, u_i, 0, 0)$ belongs to γ_{i+20}, $u_{i+4}(2)-2(0, u_{i+4}, 0, 0)$ to $-\gamma_{i+20}$, $u_i(2)-2(0, u_i, 0, 0)$ to γ_{i+24}, $u_{i+4}(2)+2(0, u_{i+4}, 0, 0)$ to $-\gamma_{i+24}$. Finally, there are eight linear functions $\gamma_{29}, \cdots, \gamma_{36}$ on $\mathfrak{H}$, with $\gamma_j(h_5)=1$, $\gamma_j(h_6)=-2$, $29\leqq j\leqq 32$, $\gamma_j(h_i)=-1/2+\delta_{j-28,i}+\delta_{4,i}$, $29\leqq j\leqq 31$, $1\leqq i\leqq 4$, $\gamma_{32}(h_i)=-1/2$, $1\leqq i\leqq 4$, and with $\gamma_j(h_5)=-1$, $\gamma_j(h_6)=2$, $33\leqq j\leqq 36$, $\gamma_j(h_i)=\gamma_{j-4}(h_i)$, $33\leqq j\leqq 36$, $1\leqq i\leqq 4$. For $1\leqq i\leqq 4$, $u_i(3)+2(0, 0, u_i, 0)$ belongs to γ_{i+28}, $u_{i+4}(3)-2(0, 0, u_{i+4}, 0)$ to $-\gamma_{i+28}$, $u_i(3)-2(0, 0, u_i, 0)$ to γ_{i+32}, $u_{i+4}(3)+2(0, 0, u_{i+4}, 0)$ to $-\gamma_{i+32}$. (These formulas are derived from (14), (34), and the formulas (2) for H_i^ϕ and H_i^ψ, $1\leqq i\leqq 4$.)

It follows as in §1 that $\mathfrak{H}$ is an abelian Cartan subalgebra of $\mathfrak{L}$ relative to which $\mathfrak{L}$ is the direct sum of $\mathfrak{H}$ and the 72 one-dimensional root-spaces $\mathfrak{L}_\alpha$, where α runs through $\pm\gamma_1, \cdots, \pm\gamma_{36}$. Since the roots α span $\mathfrak{H}^*$, one sees as before that the center of $\mathfrak{L}$ is zero. To see that $[\mathfrak{L}\mathfrak{L}]=\mathfrak{L}$ it will be enough to show that $\mathfrak{H}\subseteq[\mathfrak{L}\mathfrak{L}]$. This will follow when we show that each $[\mathfrak{L}_{-\alpha}\mathfrak{L}_\alpha]\neq 0$, hence is one-dimensional, for here the proof will show that $[\mathfrak{L}\mathfrak{L}]$ contains all h_i, $1\leqq i\leqq 6$.

Now $[u_i(1), u_{i+4}(1)]$, $1\leqq i\leqq 4$, is a derivation sending w_1 into 0 by (34), and $a(1, 2)$ into $4(a, u_i)u_{i+4}(1, 2)-4(a, u_{i+4})u_i(1, 2)$. Thus $[u_i(1), u_{i+4}(1)]=(0, 0, 0, T)$, where $aT=8(a, u_i)u_{i+4}-8(a, u_{i+4})u_i$. But this means that $T=-8H_i$, since $u_jT=0=-8u_jH_i$, $j\neq i$, $i+4$, $u_iT=-8(u_i, u_{i+4})u_i=-8u_i=-8u_iH_i$, and $u_{i+4}T=8(u_{i+4}, u_i)u_{i+4}=8u_{i+4}=-8u_{i+4}H_i$, $1\leqq i\leqq 4$. We also have $[u_i(1), (u_{i+4}, 0, 0, 0)]=R_{u_i(1,2)(u_{i+4},0,0,0)}=(\mathrm{diag}\{-2, 2, 0\})/2=-h_5$,

$[u_{i+4}(1), (u_i, 0, 0, 0)] = -h_5$, and $[(u_i, 0, 0, 0), (u_{i+4}, 0, 0, 0)] = (0, 0, 0, U)$, where $aU = 2(a, u_{i+4})u_i - 2(a, u_i)u_{i+4} = 2aH_i$, or $U = 2H_i$. Thus

$$(37) \quad [u_i(1) \pm 2(u_i, 0, 0, 0), u_{i+4}(1) \mp 2(u_{i+4}, 0, 0, 0)] = \pm 4h_5 - 8h_i,$$

$1 \leqq i \leqq 4$. By our results of §1 on $\mathfrak{D}$, namely that $h_i \in [\mathfrak{D}\mathfrak{D}]$, $1 \leqq i \leqq 4$, it follows that $h_5 \in [\mathfrak{L}\mathfrak{L}]$, and with the results of that section and (37), that $[\mathfrak{L}_{-\alpha}\mathfrak{L}_\alpha]$ is one-dimensional if α is among $\pm\gamma_1, \cdots, \pm\gamma_{20}$. Similarly, we have $[u_i(2) \pm 2(0, u_i, 0, 0), u_{i+4}(2) \mp 2(0, u_{i+4}, 0, 0)] = D_i \mp 2R_{u_i(1,3)(0,u_{i+4},0,0)} \mp 2R_{u_{i+4}(3)(0,u_i,0,0)}$, where $D_i \in \mathfrak{D}$ and $1 \leqq i \leqq 4$; also

$$[u_i(3) \pm 2(0, 0, u_i, 0), u_{i+4}(3) \mp 2(0, 0, u_{i+4}, 0)]$$
$$= E_i \mp 2R_{u_i(2,3)(0,0,u_{i+4},0)} \mp 2R_{u_{i+4}(2,3)(0,0,u_i,0)}, \quad 1 \leqq i \leqq 4, \text{ where } E_i \in \mathfrak{D}.$$

From (9), we find that

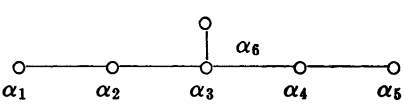
$$u_i(1, 3)(0, u_{i+4}, 0, 0) = u_{i+4}(1, 3)(0, u_i, 0, 0) = -(w_1 + w_2), u_i(2, 3)(0, 0, u_{i+4}, 0)$$
$$= u_{i+4}(2, 3)(0, 0, u_i, 0) = -w_2,$$

hence that

$$[u_i(2) \pm 2(0, u_i, 0, 0), u_{i+4}(2) \mp 2(0, u_{i+4}, 0, 0)] = D_i \pm 4(h_5 + h_6) \neq 0,$$
$$1 \leqq i \leqq 4,$$
$$[u_i(3) \pm 2(0, 0, u_i, 0), u_{i+4}(3) \mp 2(0, 0, u_{i+4}, 0)] = E_i \pm 4h_6 \neq 0.$$

This completes the proof that every $[\mathfrak{L}_{-\alpha}\mathfrak{L}_\alpha]$ is one-dimensional; we also see that $h_6 \in [\mathfrak{L}\mathfrak{L}]$, hence that $\mathfrak{H} \subseteq [\mathfrak{L}\mathfrak{L}]$ and $[\mathfrak{L}\mathfrak{L}] = \mathfrak{L}$. If the characteristic p of $\mathfrak{F}$ is not 5, and if α, β are roots relative to the Cartan subalgebra $\mathfrak{H}$ with $\beta \neq 0$, then not all $\alpha + m\beta$ (m an integer) are roots, by the fact that there are only five possible distinct values for each $\gamma(h_i)$, where γ runs through all roots (readily seen from the list of roots above). If $p = 5$, the same conclusion can be reached by checking cases. Thus $\mathfrak{L}$ and $\mathfrak{H}$ satisfy the conditions (i)–(v) of [8] for a Lie algebra of classical type.

We now show that $\mathfrak{L}$ is simple of type E_6. Namely, let $\alpha_1 = -\gamma_{17}$, the root to which $u_5(1) + 2(u_5, 0, 0, 0)$ belongs; let α_2 be the root to which $(0, 0, 0, E_{21} - E_{56})$ belongs, α_3 the root to which $(0, 0, 0, E_{32} - E_{67})$ belongs, α_4 the root to which $(0, 0, 0, E_{74} - E_{83})$ belongs, $\alpha_5 = \gamma_{32}$, the root to which $u_4(3) + 2(0, 0, u_4, 0)$ belongs, α_6 the root to which $(0, 0, 0, E_{43} - E_{78})$ belongs. These form a basis for $\mathfrak{H}^*$, and no $\alpha_i - \alpha_j$ is a root for $i \neq j$. They are a system of roots with diagram E_6:



Since the diagram is connected, $\mathfrak{L}$ is simple [8, Theorem 8]; since the system is of type E_6, it is *fundamental* in the sense of §10 of [8] (see Case 4 of the proof of Theorem 7 there), and $\mathfrak{L}$ is the (normal) simple Lie algebra of type E_6 over $\mathfrak{F}$ in the classification of [8]. With Theorem 9 of that paper, we have the

THEOREM 4. *If the base field $\mathfrak{F}$ is of characteristic different from 2 and 3, the Lie algebra $\mathfrak{L}$ of norm-skew transformations of the split exceptional Jordan algebra $\mathfrak{J}$ over $\mathfrak{F}$ is a normal simple Lie algebra of classical type E_6. Thus any simple Lie algebra of classical type E_6 over $\mathfrak{F}$ is isomorphic to $\mathfrak{L}$.*

6. The enveloping algebra of the norm-skew algebra. To apply the device of Lemma 1 of [10] to conclude the feasibility of working with exponentials for all but a few low characteristics, we next prove the following analogue of the Lemma 1 of §2:

LEMMA 2. $\mathfrak{L}^4=\mathfrak{E}(\mathfrak{J})$, *the algebra of all linear transformations of $\mathfrak{J}$.*

Proof. We take for $\mathfrak{J}$ the basis I, w_1, w_2, $u_i(j, k)$, $1\leqq i\leqq 8$, $1\leqq j<k\leqq 3$, and proceed to show that $\mathfrak{L}^4$ contains all the matrix-unit transformations relative to this basis. As in §2, $\mathfrak{L}^j\subseteq\mathfrak{L}^{j+1}$ for all $j\geqq 1$, since $\mathfrak{L}=[\mathfrak{L}\mathfrak{L}]\subseteq\mathfrak{L}^2$. With $A=(0, 0, 0, T)$ a root-vector belonging to one of $\pm\gamma_1, \cdots, \pm\gamma_{12}$, and with B also of this form, we see that $\mathfrak{L}^2$ contains AB, and among the products AB are the matrix units $E(u_i(j, k), u_{i+4}(j, k))$ and $E(u_{i+4}(j, k), u_i(j, k))$, $1\leqq i\leqq 4$, $1\leqq j<k\leqq 3$. For $(0, 0, 0, T)(0, 0, 0, U)$ sends $\operatorname{diag}\{\alpha_1, \alpha_2, \alpha_3\} + a_{12}(1, 2) + a_{13}(1, 3) + a_{23}(2, 3)$ into $(a_{12}TU)(1, 2) + (a_{13}T^{\psi}U^{\psi})(1, 3) + (a_{23}T^{\phi}U^{\phi})(2, 3)$. The result now follows by an easy computation. Still letting $A=(0, 0, 0, T)$ as before, and letting B run through the remaining 48 root-vectors, we find among the products AB scalar multiples of the following matrix units: $E(u_i(1, 2), u_j(1, 3))$, $j\in C(i)$, $1\leqq i\leqq 8$; $E(u_i(1, 2), u_j(2,3))$, $j\in C(i)$, $i\neq 4, 8$; $E(u_i(1, 2), u_j(2, 3))$, $j\in D(i)$, $i=4, 8$; $E(u_i(1, 3), u_j(1, 2))$, $j\in C(i)$, $1\leqq i\leqq 8$; $E(u_i(1, 3), u_j(2, 3))$, $j\in C'(i)$, $i\neq 4, 8$; $E(u_i(1, 3), u_j(2, 3))$, $j\in D'(i)$, $i=4, 8$; $E(u_i(2, 3), u_j(1, 2))$, $j\in C'(i)$, $1\leqq i\leqq 8$; $E(u_i(2, 3), u_j(1, 3))$, $j\in C'(i)$, $i\neq 4, 8$; $E(u_i(2, 3), u_j(1, 3))$, $j\in D'(i)$, $i=4, 8$.

Following these matrix-unit transformations by elements of the form $A=(0, 0, 0, T)$, we see that $\mathfrak{L}^3$ contains all $E(u_i(j, k), u_{i'}(j', k'))$, except possibly for the matrix units $E(u_i(j, k), u_i(j, k))$. Now for $1\leqq i\leqq 4$, we have $E(u_i(1, 2), u_{i+4}(1, 2))\in\mathfrak{L}^2$, and $E(u_i(1, 2), u_{i+4}(1, 2))(u_i(1)+2(u_i, 0, 0, 0))$ is a nonzero scalar multiple of $-E(u_i(1, 2), w_1)+E(u_i(1, 2), w_2)+E(u_i(1, 2), I)$; hence the latter is in $\mathfrak{L}^3$, as is $E(u_i(1, 2), u_{i+4}(1, 2))(u_i(1)-2(u_i, 0, 0, 0))$, a nonzero scalar multiple of $2E(u_i(1, 2), w_1)+E(u_i(1, 2), w_2)+E(u_i(1, 2), I)$. It follows that $E(u_i(1, 2), w_1)\in\mathfrak{L}^3$, $E(u_i(1, 2), w_2)+E(u_i(1, 2), I)\in\mathfrak{L}^3$, $1\leqq i\leqq 4$. A symmetric argument shows that this result holds for $1\leqq i\leqq 8$, and one likewise proves that with $j\in C(i)$, $1\leqq j\leqq 4$,

$$E(u_i(1, 2), u_j(1, 3))(u_{j+4}(2) + 2(0, u_{j+4}, 0, 0)) \in \mathfrak{L}^3,$$

yields $E(u_i(1, 2), w_1)+2E(u_i(1, 2), w_2)-E(u_i(1, 2), I)\in\mathfrak{L}^3$. With the above, we have $E(u_i(1,\ 2),\ w_2)$ and $E(u_i(1,\ 2),\ I)\in\mathfrak{L}^3$, $1\leqq i\leqq 8$. Similarly, all $E(u_i(j,\ k),\ w_1)$, $E(u_i(j,\ k),\ w_2)$, $E(u_i(j,\ k),\ I)$ are in $\mathfrak{L}^3$. Using products of the form $(u_i(1)\pm 2(u_i,\ 0,\ 0,\ 0))E(u_i(1,\ 2),\ u_{i'}(j,\ k))$, where the second factor is one of our matrix units in $\mathfrak{L}^2$, and similar products

$$(u_i(2)\ \pm\ 2(0,\ u_i,\ 0,\ 0))E(u_i(1,\ 3),\ u_{i'}(j,\ k)),$$

$$(u_i(3)\ \pm\ 2(0,\ 0,\ u_i,\ 0))E(u_i(2,\ 3),\ u_{i'}(j,\ k)),$$

we find that $\mathfrak{L}^3$ contains all the matrix units $E(w_1,\ u_i(j,\ k))$, $E(w_2,\ u_i(j,\ k))$, $E(I,\ u_i(j,\ k))$. From the fact that $E(u_i(j,\ k),\ u_{i+4}(j,\ k))$ and $E(u_{i+4}(j,\ k),\ u_i(j,\ k))$ are in $\mathfrak{L}^2$ for $1\leqq i\leqq 4$, we see that $\mathfrak{L}^4$ contains $E(u_i(j,\ k),\ u_i(j,\ k))$, $1\leqq i\leqq 8$, $1\leqq j<k\leqq 3$.

Now it has been shown that for $w=w_1,\ w_2,\ I$, we have $E(w,\ u_i(j,\ k))\in\mathfrak{L}^3$. Thus $E(w,u_1(1,2))(u_5(1)\pm 2(u_5,0,0,0))$ and $E(w,u_1(1,3))(u_5(2)+2(0,u_5,0,0))$ are in $\mathfrak{L}^4$; it follows that $\mathfrak{L}^4$ contains $-E(w,\ w_1)+E(w,w_2)+E(w,\ I)$, $-2E(w,\ w_1)-E(w,\ w_2)-E(w,\ I)$, $-E(w,\ w_1)-2E(w,\ w_2)+E(w,\ I)$, and therefore that all three of $E(w,\ w_1)$, $E(w,\ w_2)$, $E(w,\ I)$ are in $\mathfrak{L}^4$. Thus all matrix units are in $\mathfrak{L}^4$, and $\mathfrak{L}^4=\mathfrak{E}(\mathfrak{J})$, as asserted.

It follows that the only linear transformations of $\mathfrak{J}$ which commute with all elements of $\mathfrak{L}$ are the scalars. For each of the 72 root-vectors $E_\alpha\in\mathfrak{L}$ as above we have $E_\alpha^2=0$, hence $\exp(\lambda E_\alpha)=I+\lambda E_\alpha$. More generally, if E_α is a root-vector relative to any standard Cartan subalgebra of $\mathfrak{L}$, we have $(\mathrm{ad}(E_\alpha))^3=0$, and E_α is nilpotent. From this and the fact that $\mathfrak{L}^4=\mathfrak{E}(\mathfrak{J})$ it follows by Lemma 1 of [10] that $E_\alpha^9=0$. Since we wish to use the result (7) of [10] for trilinear forms, we are led to assume that the characteristic of $\mathfrak{F}$ is different from 2, 3, 5, 7, 11, 13, 17, 19, 23. In the sequel we assume this restriction on the characteristic.

7. **Automorphisms of the norm-skew algebra.** From §1 of [10], it now follows that the group of invariant automorphisms $\mathcal{I}(\mathfrak{L})$ of $\mathfrak{L}$ is a (normal) subgroup of the group of all mappings $X\rightarrow A^{-1}XA$, where A is a nonsingular linear transformation of $\mathfrak{J}$ such that $(xA,\ yA,\ zA)=(x,\ y,\ z)$ for all $x,\ y,\ z\in\mathfrak{J}$. In fact, if A is any linear transformation of $\mathfrak{J}$ such that $(xA,\ yA,\ zA)=(x,\ y,\ z)$ for all $x,\ y,\ z\in\mathfrak{J}$, then A is nonsingular; for if $xA=0$, then $0=(xA,\ IA,\ IA)=(x,\ I,\ I)=4\,\mathrm{Tr}(x)$, or $x\in\mathfrak{J}'$. If $y\in\mathfrak{J}'$, then $0=(xA,\ yA,\ IA)=(x,\ y,\ I)=-\mathrm{Tr}(x\cdot y)=-(x,\ y)$. Thus $x\in\mathfrak{J}'$ and $(x,\ y)=0$ for all $y\in\mathfrak{J}'$. But we have seen that $(a,\ b)$ is nondegenerate on $\mathfrak{J}$, and therefore also on the orthogonal complement $\mathfrak{J}'$ of the nonisotropic subspace $\mathfrak{F}I$. Hence $x=0$, and A is nonsingular.

Moreover, by §6, if $X=A^{-1}XA$ for all $X\in\mathfrak{L}$, then $A=\lambda I$, $\lambda\in\mathfrak{F}$, and if for this A, $(xA,\ yA,\ zA)=(x,\ y,\ z)$ for all $x,\ y,\ z\in\mathfrak{J}$, then $\lambda^3=1\in\mathfrak{F}$. Since the group $\mathfrak{G}$ of all linear transformations A of $\mathfrak{J}$ such that $(xA,\ yA,\ zA)=(x,\ y,\ z)$ for all $x,\ y,\ z$ contains the $\exp(E_\alpha)=I+E_\alpha$, where E_α runs through the root-vectors relative to $\mathfrak{H}$ in $\mathfrak{L}$, and since every transformation of $\mathfrak{J}$

which commutes with all these commutes with all E_α, hence with all of $\mathfrak{E}(\mathfrak{J})$ by §6 and the fact that these E_α generate $\mathfrak{L}$, we see that the center $\mathfrak{Z}$ of $\mathfrak{G}$ consists of the mappings λI, $\lambda \in \mathfrak{F}$, $\lambda^3 = 1$. Thus the group of invariant automorphisms of $\mathfrak{L}$ may be regarded as a (normal) subgroup of the group $\mathfrak{G}/\mathfrak{Z}$.

If T is a linear transformation of $\mathfrak{J}$, we denote by T^* its adjoint with respect to the trace form $(x, y) = \mathrm{Tr}(x \cdot y)$: $(xT, y) = (x, yT^*)$ for all $x, y \in \mathfrak{J}$. Then $D^* = -D$ for all $D \in \mathfrak{D}$ by (29) and $R_a^* = R_a$ for all $a \in \mathfrak{J}$ by the associativity of the form. Since $\mathfrak{L} = \mathfrak{D} \oplus \mathfrak{R}_{\mathfrak{J}'}$, the mapping $T \rightarrow -T^*$ maps $\mathfrak{L}$ onto itself, is the identity on $\mathfrak{D}$, and is the negative of the identity on $\mathfrak{R}_{\mathfrak{J}'}$. Moreover it is evidently an automorphism of $\mathfrak{L}$. Next we show that this automorphism is not of the form $X \rightarrow A^{-1}XA$, where A is nonsingular in $\mathfrak{E}(\mathfrak{J})$. For then we should have $DA = AD$ for all $D \in \mathfrak{D}$, $R_a A = -AR_a$ for all $a \in \mathfrak{J}'$. Therefore if $a \in \mathfrak{J}'$, $2aA = (I \cdot a)A = (IR_a)A = -(IA)R_a = -(IA) \cdot a = -aR_{IA}$. Thus $aA = -aR_{IA}/2$ whenever $a \in \mathfrak{J}'$. It is also clear that $IA = IR_{IA}/2$. If we set $b = IA/2$, we have $(\lambda I + a)A = \lambda IR_b - aR_b$, where $\lambda \in \mathfrak{F}$, $a \in \mathfrak{J}'$. If $D \in \mathfrak{D}$, then $[AD] = 0$, so that $0 = (\lambda I + a)[AD] = \lambda I[AD] + a[AD]$. With $a = 0$, we see from $ID = 0$ that $\lambda IAD = 0 = 2\lambda bD$ for all $\lambda \in \mathfrak{F}$, hence that $bD = 0$ for all $D \in \mathfrak{D}$. If $b = \beta I + b'$, $\beta \in \mathfrak{F}$, $b' \in \mathfrak{J}'$, then $b'D = 0$ for all $D \in \mathfrak{D}$. But we have seen in §2 that $\mathfrak{D}^4 = \mathfrak{E}(\mathfrak{J}')$, so that $b' = 0$, $b = \beta I$, $(\lambda I + a)A = 2\beta\lambda I - 2\beta a$. Next, taking $\lambda = 0$, $a = w_1 \in \mathfrak{J}'$, we have $(w_1 \cdot w_1)A = w_1 R_{w_1} A = -w_1 A R_{w_1} = 2\beta w_1 R_{w_1} = 2\beta(w_1 \cdot w_1)$. However, $w_1 \cdot w_1 = 2(2I + w_1 + 2w_2)/3$, so that $(w_1 \cdot w_1)A = 4\beta(2I - w_1 - 2w_2)/3 \neq 2\beta(w_1 \cdot w_1)$ unless $\beta = 0$. This implies $A = 0$, contrary to the assumption that A is nonsingular. Hence the automorphism $X \rightarrow -X^*$ is not of the form $X \rightarrow A^{-1}XA$.

In case $\mathfrak{F}$ is algebraically closed, we have seen in [9] that the index of the group $\mathfrak{g}(\mathfrak{L})$ in the full automorphism group $\mathfrak{A}(\mathfrak{L})$ is either 1 or 2. Since the automorphism $X \rightarrow -X^*$ is not in $\mathfrak{g}(\mathfrak{L})$ by the above, this index is 2. Since $X \rightarrow -X^*$ is not in the group of automorphisms $X \rightarrow A^{-1}XA$, $A \in \mathfrak{G}$, and since this group contains $\mathfrak{g}(\mathfrak{L})$, we see that it must coincide with $\mathfrak{g}(\mathfrak{L})$. Thus every automorphism of $\mathfrak{L}$ is either of the form $X \rightarrow A^{-1}XA$, where $A \in \mathfrak{G}$, or of the form $X \rightarrow -A^{-1}X^*A$, where $A \in \mathfrak{G}$. The group $\mathfrak{g}(\mathfrak{L})$ is isomorphic to $\mathfrak{G}/\mathfrak{Z}$, and consists of all automorphisms of $\mathfrak{L}$ of the former of these types.

If $\mathfrak{F}$ is not algebraically closed, let $\mathfrak{K}$ be its algebraic closure, and consider $\mathfrak{L}_{\mathfrak{K}}$, which as in previous cases is the Lie algebra of type E_6 formed from the split exceptional Jordan algebra $\mathfrak{J}_{\mathfrak{K}}$ by the above process. Every automorphism σ of $\mathfrak{L}$ extends to a unique automorphism of $\mathfrak{L}_{\mathfrak{K}}$, which is either of the form $X \rightarrow A^{-1}XA$ or of the form $X \rightarrow -A^{-1}X^*A$, where A preserves the norm form in $\mathfrak{J}_{\mathfrak{K}}$ and where X^* is the adjoint of X with respect to the trace form in $\mathfrak{J}_{\mathfrak{K}}$. Since the trace and norm forms in $\mathfrak{J}_{\mathfrak{K}}$ extend those in $\mathfrak{J}$, we see that every automorphism σ of $\mathfrak{L}$ is either of the form $X \rightarrow A^{-1}XA$ or of the form $X \rightarrow -A^{-1}X^*A$, where A is a linear transformation of $\mathfrak{J}_{\mathfrak{K}}$ such that $(xA, yA, zA) = (x, y, z)$ for all $x, y, z \in \mathfrak{J}$. As in §3, it follows that either $X^\sigma = B^{-1}XB$ for all

$X \in L$ or $X^\sigma = -B^{-1}X^*B$, where B is a nonsingular linear transformation of $\mathfrak{J}$. As before, we have $B_{\mathfrak{K}}A^{-1} = \lambda I$, $\lambda \in \mathfrak{K}$, $B_{\mathfrak{K}} = \lambda A$, so that for $x, y, z \in \mathfrak{J}$, $(xB, yB, zB) = (xB_{\mathfrak{K}}, yB_{\mathfrak{K}}, zB_{\mathfrak{K}}) = \lambda^3(xA, yA, zA) = \lambda^3(x, y, z)$. Since B is nonsingular, some $(xB, yB, zB) \neq 0$, and $\lambda^3 \in \mathfrak{F}$. Thus there is a nonzero $\beta \in \mathfrak{F}$ such that $(xB, yB, zB) = \beta(x, y, z)$ for all $x, y, z \in \mathfrak{J}$. If we let $\mathfrak{B}$ be the group of all linear transformations of $\mathfrak{J}$ with this property, we see that every automorphism of $\mathfrak{L}$ is either of the form $X \to B^{-1}XB$, $B \in \mathfrak{B}$, or of the form $X \to -B^{-1}X^*B$, $B \in \mathfrak{B}$ (but not both, by the second previous paragraph). The center of $\mathfrak{B}$ commutes with all elements of $\mathfrak{G}$, hence consists of all nonzero scalars λI, $\lambda \in \mathfrak{F}$, and the automorphism group $\mathfrak{A}(\mathfrak{L})$ is the semi-direct product of $\mathfrak{B}/\mathfrak{F}^*I$ by the cyclic group of order 2 generated by $X \to -X^*$. To summarize:

Theorem 5. *If the characteristic of $\mathfrak{F}$ is zero or a prime greater than 23, and if $\mathfrak{L}$ is the Lie algebra of all norm-skew transformations of the split exceptional Jordan algebra $\mathfrak{J}$ over $\mathfrak{F}$, then every automorphism of $\mathfrak{L}$ is of one and only one of the forms $X \to B^{-1}XB$, $X \to -B^{-1}X^*B$, where X^* is the adjoint of X with respect to the bilinear form $(x, y) = \mathrm{Tr}(x \cdot y)$ in $\mathfrak{J}$, and where $B \in \mathfrak{B}$, the group of all linear transformations of $\mathfrak{J}$ such that for some $\beta \neq 0$ in $\mathfrak{F}$, $(xB, yB, zB) = \beta(x, y, z)$ for all $x, y, z \in \mathfrak{J}$. If $\mathfrak{F}$ is algebraically closed, B may be taken from the subgroup $\mathfrak{G}$ of $\mathfrak{B}$ consisting of those $B \in \mathfrak{B}$ for which $\beta = 1$, and the transformations $X \to B^{-1}XB$, $B \in \mathfrak{G}$, constitute the group of invariant automorphisms of $\mathfrak{L}$. The automorphism group of $\mathfrak{L}$ is thus the semi-direct product of the group $\mathfrak{B}/\mathfrak{F}^*I$ and a cyclic group of order 2; if $\mathfrak{F}$ is algebraically closed, we can replace $\mathfrak{B}/\mathfrak{F}^*I$ by the isomorphic group $\mathfrak{G}/\mathfrak{Z}$, where $\mathfrak{Z} = \{\lambda I \mid \lambda^3 = 1\}$.*

8. Generation of the group $\mathfrak{G}$ of norm-preserving transformations. Finally we show that $\mathscr{I}(\mathfrak{L}) \cong \mathfrak{G}/\mathfrak{Z}$ in all cases by showing that $\mathfrak{G}/\mathfrak{Z}$ is the subgroup G' of $\mathscr{I}(\mathfrak{L})$ studied by Chevalley [2], that is, the subgroup generated by all $\exp(\mathrm{ad}(E_\alpha))$, where E_α runs through all root-vectors belonging to nonzero roots in *one* standard Cartan decomposition of $\mathfrak{L}$. The simplicity of $\mathfrak{G}/\mathfrak{Z}$ (and of $\mathscr{I}(\mathfrak{L})$) will then follow by Chevalley's results. (In unpublished work, Jacobson has proved the simplicity of $\mathfrak{G}/\mathfrak{Z}$ by more direct methods.) The construction of Chevalley starts with a simple complex Lie algebra of type E_6 and obtains a standard Cartan decomposition of type E_6 of a Lie algebra of classical type over the field $\mathfrak{F}$. Then this algebra is isomorphic to our algebra $\mathfrak{L}$, and under the isomorphism the generators of the group G' acting in Chevalley's algebra correspond to mappings $\exp(\mathrm{ad}(E_\alpha))$ in $\mathfrak{L}$, where E_α runs through the root-vectors relative to a standard Cartan subalgebra $\mathfrak{H}_0$ of $\mathfrak{L}$, a fundamental system of roots of $\mathfrak{L}$ relative to $\mathfrak{H}_0$ being of type E_6. It follows by Theorem 9 of [8] that there is an automorphism σ of $\mathfrak{L}$ mapping the Cartan subalgebra $\mathfrak{H}$ used in our decomposition of §5 onto the new subalgebra $\mathfrak{H}_0$. Since $\mathfrak{H}$ is mapped into $\mathfrak{H}$ by

$$X \to -X^* \qquad (-h_i^* = h_i,\ 1 \leqq i \leqq 4;\ -h_j^* = -h_j,\ j = 5, 6),$$

we may assume that σ is of the form $X \to A^{-1}XA$, $A \in \mathfrak{B}$. The system of root-vectors $\{E'_\alpha\}$ relative to $\mathfrak{H}_0$ thus consists of all scalar multiples of the elements $A^{-1}E_\alpha A$, where E_α runs through the 72 root-vectors relative to $\mathfrak{H}$ as listed in §5. Now $\exp(\operatorname{ad}(A^{-1}E_\alpha A))$ sends $X \in \mathfrak{L}$ into

$$A^{-1}(\exp(E_\alpha))^{-1}AXA^{-1}(\exp(E_\alpha))A,$$

i.e., conjugates X by an element of $A^{-1}GA$, where G is the subgroup of $\mathfrak{G}$ generated by all $\exp(\lambda E_\alpha)$, $\lambda \in \mathfrak{F}$, E_α a root-vector relative to $\mathfrak{H}$. We show $G = \mathfrak{G}$; then, since $\mathfrak{G}$ is an invariant subgroup of $\mathfrak{B}$, it will follow that $A^{-1}GA = \mathfrak{G}$, thus that $G' \cong \mathfrak{G}/\mathfrak{Z} \cong \mathfrak{g}(\mathfrak{L})$.

We have seen that G contains the $\exp(\lambda(0, 0, 0, T))$, where $(0, 0, 0, T) \in \mathfrak{D}$ belongs to each of the roots $\pm\gamma_1, \cdots, \pm\gamma_{12}$ relative to $\mathfrak{H}$. Now we compute the effect on the general element

$$a = \operatorname{diag}\{\alpha_1, \alpha_2, \alpha_3\} + a_{12}(1, 2) + a_{13}(1, 3) + a_{23}(2, 3)$$

of $\mathfrak{J}$ of the exponentials of the remaining root-vectors:

$$\begin{aligned}
a \exp(\lambda(u_i(1) + 2(u_i, 0, 0, 0))) &= a + \lambda[4(u_i, a_{12}) \operatorname{diag}\{0, 1, 0\} \\
&\quad + 2\alpha_1 u_i(1, 2) + 2(\bar{u}_i a_{13})(2, 3)]; \\
a \exp(\lambda(u_i(1) - 2(u_i, 0, 0, 0))) &= a + \lambda[4(u_i, a_{12}) \operatorname{diag}\{1, 0, 0\} \\
&\quad + 2\alpha_2 u_i(1, 2) + 2(u_i a_{23})(1, 3)]; \\
a \exp(\lambda(u_i(2) + 2(0, u_i, 0, 0))) &= a + \lambda[4(u_i, a_{13}) \operatorname{diag}\{0, 0, 1\} \\
&\quad + 2\alpha_1 u_i(1, 3) + 2(\bar{a}_{12} u_i)(2, 3)]; \\
a \exp(\lambda(u_i(2) - 2(0, u_i, 0, 0))) &= a + \lambda[4(u_i, a_{13}) \operatorname{diag}\{1, 0, 0\} \\
&\quad + 2(u_i \bar{a}_{23})(1, 2) + 2\alpha_3 u_i(1, 3)]; \\
a \exp(\lambda(u_i(3) + 2(0, 0, u_i, 0))) &= a + \lambda[4(u_i, a_{23}) \operatorname{diag}\{0, 0, 1\} \\
&\quad + 2(a_{12} u_i)(1, 3) + 2\alpha_2 u_i(2, 3)]; \\
a \exp(\lambda(u_i(3) - 2(0, 0, u_i, 0))) &= a + \lambda[4(u_i, a_{23}) \operatorname{diag}\{0, 1, 0\} \\
&\quad + 2(a_{13} \bar{u}_i)(1, 2) + 2\alpha_3 u_i(2, 3)].
\end{aligned}$$

NOTE. If we denote by $T_{ij}(u)$, $u \in \mathfrak{C}$, $i \neq j$, that 3 by 3 $\mathfrak{C}$-matrix with 1 in all diagonal positions and with u in the (i, j)-position and zero elsewhere, then for $a \in \mathfrak{J}$, the element $(T_{ij}(u)a)T_{ji}(\bar{u}) = T_{ij}(u)(aT_{ji}(\bar{u}))$, defined by ordinary matrix multiplication, is again in $\mathfrak{J}$. The mappings $a \to (T_{ij}(u)a)T_{ji}(\bar{u})$ of $\mathfrak{J}$, as u runs through all scalar multiples of basis elements $u_1, \cdots, u_8$ for $\mathfrak{C}$, are just the mappings listed above. In their more general form, they have been utilized by Jacobson in unpublished work on the norm-preserving groups.

By a simple calculation,

$$\exp(\lambda(u_i(1) + 2(u_i, 0, 0, 0))) \exp(-\lambda(u_i(1) - 2(u_i, 0, 0, 0))) = \exp(4\lambda(u_i, 0, 0, 0));$$

$$\exp(\lambda(u_i(2) + 2(0, u_i, 0, 0))) \exp(-\lambda(u_i(2) - 2(0, u_i, 0, 0))) = \exp(4\lambda(0, u_i, 0, 0));$$

$$\exp(\lambda(u_i(3) + 2(0, 0, u_i, 0))) \exp(-\lambda(u_i(3) - 2(0, 0, u_i, 0))) = \exp(4\lambda(0, 0, u_i, 0)).$$

Since the transformations $\exp(\mu(u_i, 0, 0, 0))$, $\exp(\mu(0, u_i, 0, 0))$, $\exp(\mu(0, 0, \mu_i, 0))$, together with the $\exp(\lambda(0, 0, 0, T))$, where $(0, 0, 0, T)$ is a root-vector, generate the full group $\mathfrak{A}(\mathfrak{J})$ by §4, we have $\mathfrak{A}(\mathfrak{J}) \subseteq G$.

As has been observed by Jacobson in a more general setting [6], an element T of the norm-preserving group $\mathfrak{G}$ is in $\mathfrak{A}(\mathfrak{J})$ if and only if $IT = I$. Since $\mathfrak{A}(\mathfrak{J}) \subseteq G \subseteq \mathfrak{G}$ by the above, this may be verified for our special case by assuming $T \in \mathfrak{G}$, $IT = I$, and by showing that it follows that $T \in \mathfrak{A}(\mathfrak{J})$. From $(I, I, a) = (IT, IT, aT) = (I, I, aT)$ we find that $\mathrm{Tr}(aT) = \mathrm{Tr}(a)$ for all $a \in \mathfrak{J}$. Together with $(I, a, b) = (IT, aT, bT) = (I, aT, bT)$, this yields $(aT, bT) = (a, b)$ for all $a, b \in \mathfrak{J}$, hence that $\tau(aT, bT, cT) = \tau(a, b, c)$ for all $a, b, c \in \mathfrak{J}$, by the fact that $\tau(a, b, c) = (a, b, c) + \mathrm{Tr}(a)(b, c) + \mathrm{Tr}(b)(a, c) + \mathrm{Tr}(c)(a, b) - 2\,\mathrm{Tr}(a)\mathrm{Tr}(b)\mathrm{Tr}(c)$. Thus $((aT \cdot bT), cT) = \tau(aT, bT, cT) = \tau(a, b, c) = (a \cdot b, c) = ((a \cdot b)T, cT)$ for all $a, b, c \in \mathfrak{J}$. Since cT runs through $\mathfrak{J}$, we have $(a \cdot b)T = aT \cdot bT$ for all a, b, by the nondegeneracy of (x, y). Thus T is an automorphism.

It therefore follows that to show $G = \mathfrak{G}$, it suffices to show that if $T \in \mathfrak{G}$, then a sequence of multiplications by elements of G can transform T into an element U of $\mathfrak{G}$ with $IU = I$. Then $U \in \mathfrak{A}(\mathfrak{J}) \subseteq G$, and so $T \in G$. Now $(I, I, I) = 12 \in \mathfrak{F}$, so that $(IT, IT, IT) = 12$. If $b = IT$, we wish to show there is a $V \in G$ such that $bV = I$; we prove that this is the case for all $b \in \mathfrak{J}$ with $(b, b, b) = 12$. Thus let $b = \mathrm{diag}\{\beta_1, \beta_2, \beta_3\} + b_{12}(1, 2) + b_{13}(1, 3) + b_{23}(2, 3) \in J$, $(b, b, b) = 12$.

First we note that $\beta_1 = 0$, $b_{12} = b_{13} = 0$, is impossible, since these conditions imply $(b, b, b) = 0$. Now if $b_{12} = b_{13} = 0$, then $\beta_1 \neq 0$, and we may operate on b with $\exp(u_1(1) + 2(u_1, 0, 0, 0))$ to obtain an element c of norm 12 with $c_{12} \neq 0$. Thus we may assume that either $b_{12} \neq 0$ or $b_{13} \neq 0$. If $b_{12} \neq 0$ then for some i, $(b_{12}, u_i) \neq 0$, and we may operate on b with $\exp(\lambda(u_i(1) - 2(u_i, 0, 0, 0)))$, where λ is so chosen that $\beta_1 + 4\lambda(b_{12}, u_i) = 1$, to make β_1 into 1. If $b_{13} \neq 0$, a similar procedure may be applied with an $\exp(\lambda(u_i(2) - 2(0, u_i, 0, 0)))$. Thus we may assume $\beta_1 = 1$. Now if $b_{12} = \sum \alpha_i u_i$, $b_{13} = \sum \gamma_i u_i$, we operate on b successively with the generators $\exp(-\alpha_i(u_i(1) + 2(u_i, 0, 0, 0))/2)$, $1 \leqq i \leqq 8$, so that b_{12} is replaced by 0, and the condition $\beta_1 = 1$ is unaffected. Now the properties $b_{12} = 0$, $\beta_1 = 1$ are unchanged when we operate successively with the generators $\exp(-\gamma_i(u_i(2) + 2(0, u_i, 0, 0))/2)$, $1 \leqq i \leqq 8$, and the result now has

$\beta_1=1$, $b_{12}=0=b_{13}$. Thus we may assume b has this form: $b=\text{diag}\{1, \beta_2, \beta_3\}+b_{23}(2, 3)$.

Now if $\beta_2=0$ and $b_{23}=0$, then $(b, b, b)=0$, which is impossible. If $b_{23}\neq 0$, we choose u_i so that $(u_i, b_{23})\neq 0$, then choose λ so that $\beta_2+4\lambda(u_i, b_{23})=1$. Then operating on b with $\exp(\lambda(u_i(3)-2(0, 0, u_i, 0)))$ leaves $\beta_1=1$, $b_{12}=0=b_{13}$, and replaces β_2 by 1. If $b_{23}=0$ then $\beta_2\neq 0$, and operating on b with $\exp(u_1(3)+2(0, 0, u_1, 0))$ leaves $\beta_1=1$, $b_{12}=0=b_{13}$, and replaces b_{23} by $2\beta_2 u_1\neq 0$. Then we can replace β_2 by 1 as above, so may assume $b=\text{diag}\{1, 1, \beta_3\}+b_{23}(2, 3)$. If $b_{23}=\sum\delta_i u_i$, successive application of the

$$\exp(-\delta_i(u_i(3)+2(0, 0, u_i, 0))/2), \qquad 1\leqq i\leqq 8,$$

leaves unchanged $\beta_1=1$, $\beta_2=1$, $b_{12}=0$, $b_{13}=0$, and replaces b_{23} by 0. We have therefore transformed b by generators of G into $\text{diag}\{1, 1, \beta_3\}$, and may assume $b=\text{diag}\{1, 1, \beta_3\}$, $(b, b, b)=12$. But $(b, b, b)=4(2+\beta_3^3)-6(2+\beta_3)(2+\beta_3^2)+2(2+\beta_3)^3=12\beta_3$. Hence $\beta_3=1$, and b has been transformed into the identity as required. We have thus completed the proof of the following result.

THEOREM 6. *If the base field $\mathfrak{F}$ is not of characteristic 2 or 3, the transformations $I+\lambda E_\alpha$, where E_α runs through a set of root-vectors relative to the Cartan subalgebra $\mathfrak{H}$ of §5 of the Lie algebra $\mathfrak{L}$ of norm-skew transformations of the split exceptional Jordan algebra $\mathfrak{J}$ generate the group $\mathfrak{G}$ of norm-preserving transformations of $\mathfrak{J}$. If the characteristic of $\mathfrak{F}$ is not a prime between 2 and 23, this result establishes that the group G' of Chevalley formed over $\mathfrak{F}$ from a simple complex Lie algebra of type E_6 may be identified with the group $\mathfrak{I}(\mathfrak{L})$ of invariant automorphisms of $\mathfrak{L}$, and is also isomorphic to $\mathfrak{G}/\mathfrak{Z}$, where $\mathfrak{Z}$ is the center of $\mathfrak{G}$; the isomorphism is induced by the homomorphism $A\to I_A$: $X\to A^{-1}XA$ of $\mathfrak{G}$ onto $\mathfrak{I}(\mathfrak{L})$.*

Finally, we note that the mapping $B\to\beta(B)$ of $\mathfrak{B}$ into the group $\mathfrak{F}^*$ defined by $\beta(B)(x, y, z)=(xB, yB, zB)$ for all $x, y, z\in\mathfrak{J}$ is a homomorphism with kernel $\mathfrak{G}$. In fact, $\mathfrak{B}$ is mapped onto $\mathfrak{F}^*$; for if $x=\text{diag}\{\xi_1, \xi_2, \xi_3\}+x_{12}(1, 2)+x_{13}(1, 3)+x_{23}(2, 3)\in\mathfrak{J}$, then $(x, x, x)=12((x_{12}x_{23}, x_{13})-\xi_1(x_{23}, x_{23})-\xi_2(x_{13}, x_{13})-\xi_3(x_{12}, x_{12})+\xi_1\xi_2\xi_3)$, so that if $0\neq\beta\in\mathfrak{F}$ and $x_{12}=\sum\alpha_i u_i$, $x_{13}=\sum\beta_i u_i$, $x_{23}=\sum\gamma_i u_i$, the transformation B of $\mathfrak{J}$ defined as follows has $(xB, xB, xB)=\beta(x, x, x)$ for all $x\in\mathfrak{J}$:

$$xB=\text{diag}\{\beta\xi_1, \xi_2, \xi_3\}+y_{12}(1, 2)+y_{13}(1, 3)+y_{23}(2, 3),$$

where

$$y_{12}=\beta\sum_{i\in C(4)}\alpha_i u_i+\sum_{i\in D(4)}\alpha_i u_i,$$

$$y_{13}=\sum_{i\in C'(4)}\beta_i u_i+\beta\sum_{i\in D'(4)}\beta_i u_i,$$

$$y_{23}=\sum_{i\neq 4,8}\gamma_i u_i+\beta^{-1}\gamma_4 u_4+\beta\gamma_8 u_8.$$

By polarization $(xB, yB, zB)=\beta(x, y, z)$ for all $x, y, z\in\mathfrak{J}$, so that $B\in\mathfrak{B}$ and $\beta(B)=\beta$. Thus we conclude $\mathfrak{B}/\mathfrak{G}\cong\mathfrak{F}^*$. If we denote by $\mathfrak{F}^{*3}$ the group of third powers of elements of $\mathfrak{F}^*$, then following $B\to\beta(B)$ by the canonical homomorphism of $\mathfrak{F}^*$ onto $\mathfrak{F}^*/\mathfrak{F}^{*3}$ gives a homomorphism of $\mathfrak{B}$ onto $\mathfrak{F}^*/\mathfrak{F}^{*3}$. The kernel evidently contains the kernel $\mathfrak{F}^*I$ of the homomorphism $B\to I_B$ of $\mathfrak{B}$ onto the group $I_{\mathfrak{B}}$ of automorphisms of $\mathfrak{L}$ of the form $X\to B^{-1}XB$, $B\in\mathfrak{B}$. Hence a homomorphism $I_B\to\beta(B)\mathfrak{F}^{*3}$ of $I_{\mathfrak{B}}$ onto $\mathfrak{F}^*/\mathfrak{F}^{*3}$ is induced. If $\beta(B)=\lambda^3\in\mathfrak{F}^{*3}$, then $I_B=I_{(\lambda^{-1}B)}$, and $\lambda^{-1}B\in\mathfrak{G}$; thus the kernel of $I_B\to\beta(B)\mathfrak{F}^{*3}$ is contained in the group $I_{\mathfrak{G}}=\mathfrak{I}(\mathfrak{L})$ of invariant automorphisms. The reverse inclusion is trivial, so we see that $I_B/I_{\mathfrak{G}}=I_{\mathfrak{B}}/\mathfrak{I}(\mathfrak{L})$ is isomorphic to $\mathfrak{F}^*/\mathfrak{F}^{*3}$. Thus the full automorphism group has the normal series $\mathfrak{A}(\mathfrak{L})\supset I_{\mathfrak{B}}\supset\mathfrak{I}(\mathfrak{L})\supset(I)$, with $\mathfrak{A}(\mathfrak{L})/I_{\mathfrak{B}}\cong Z_2$, $I_{\mathfrak{B}}/\mathfrak{I}(\mathfrak{L})\cong\mathfrak{F}^*/\mathfrak{F}^{*3}$, and $\mathfrak{I}(\mathfrak{L})\cong\mathfrak{G}/\mathfrak{Z}$, a simple group.

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